Asymptotically additive families of functions and a physical equivalence problem for flows

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Abstract

We show that additive and asymptotically additive families of continuous functions with respect to suspension flows are physically equivalent. In particular, the equivalence result holds for hyperbolic flows. We also obtain an equivalence relation for expansive flows. Moreover, we show how this equivalence result can be used to extend the nonadditive thermodynamic formalism and multifractal analysis for flows.

1 Introduction

Let X be a topological space and $T : X \to X$ a map. A sequence of functions $(f_n)_{n\geq 1}$ is *asymptotically additive* with respect to T if for each $\varepsilon > 0$ there exists a function $f : X \to \mathbb{R}$ such that

$$\limsup_{n\to\infty} \frac{1}{n}\|f_n - S_n f\|_\infty < \varepsilon,$$

where $S_n f := \sum_{k=0}^{n-1} f \circ T^k$ and $\|\cdot\|_\infty$ is the supremum norm. Notice that the sequence $(S_n f)_{n\geq 1}$ is additive with respect to T, that is,

$$S_{m+n} f(x) = S_m f(x) + S_n f(T^m(x)) \quad \text{for all } x \in X \text{ and } m, n \geq 1.$$

A sequence $\mathcal{F} = (f_n)_{n\geq 1}$ is *almost additive* with respect to T if there exists $C > 0$ such that

$$-C + f_m(x) + f_n(T^m(x)) \leq f_{m+n}(x) \leq f_m(x) + f_n(T^m(x)) + C$$

for every $x \in X$ and all $m, n \geq 1$. It was showed in [FH10] that almost additive sequences are in fact asymptotically additive.

Inspired by statistical mechanics, as in [Cun20], we say that two nonadditive sequences of functions $\mathcal{F} := (f_n)_{n\geq 1}$ and $\mathcal{G} := (g_n)_{n\geq 1}$ are *physically equivalent*, or $\mathcal{F}$ is *physically equivalent* to $\mathcal{G}$, if

$$\lim_{n\to\infty} \frac{1}{n}\|f_n - g_n\|_\infty = 0.$$

Surprisingly, N. Cuneo showed in [Cun20] that asymptotically additive sequences are physically equivalent to additive sequences with respect to a continuous map. This result has a direct impact in the study of nonadditive thermodynamic formalism and multifractal analysis for discrete-time dynamical systems (see [Cun20] and references within).

Motivated by the nonadditive thermodynamic formalism and multifractal analysis for flows, and inspired by Cuneo's equivalence theorem, in this paper we investigate the same equivalence problem in the case of continuous flows.

Let $\Phi = (\phi_t)_{t\in\mathbb{R}}$ be a continuous flow on a topological space X. A family $a = (a_t)_{t\geq 0}$ of functions $a_t\colon X \to \mathbb{R}$ is said to be *almost additive* with respect to Φ on X if there exists a constant $C > 0$ such that

$$-C + a_t + a_s \circ \phi_t \leq a_{t+s} \leq a_t + a_s \circ \phi_t + C$$

for every $t, s \geq 0$.

We also say that a family of functions $a = (a_t)_{t\geq 0}$ is *asymptotically additive* with respect to Φ on X if for each $\varepsilon > 0$ there exists a function $b_\varepsilon : X \to \mathbb{R}$ such that

$$\limsup_{t\to\infty} \frac{1}{t}\left\| a_t - \int_0^t (b_\varepsilon \circ \phi_s) ds \right\|_\infty \leq \varepsilon.$$

Proceeding as in [FH10], one can see that every almost additive family of functions is asymptotically additive.

Following the definition for discrete-time dynamical systems, we say that two families of functions $a = (a_t)_{t\geq 0}$ and $b = (b_t)_{t\geq 0}$ are *physically equivalent*, or a is *physically equivalent* to b, with respect to the flow Φ if

$$\lim_{t\to\infty} \frac{1}{t}\|a_t - b_t\|_\infty = 0.$$

We observe that physically equivalent almost additive families have the same topological pressure and share the same equilibrium measures (see [BH21a]). Moreover, they also share the same level sets and the same maximizing measures in the sense of ergodic optimization (see for example [BD09], [BH21b], [BHVZ21], [HLMXZ19] and [MSV20]).

In the present work, we show that asymptotically additive families of continuous functions are physically equivalent to additive families of continuous functions with respect to suspension flows, and in particular, hyperbolic flows and expansive flows admitting a measure of full support. As in the discrete-time case, this physical equivalence result for flows has the potential to facilitate and simplify many extensions of the nonadditive thermodynamic formalism, multifractal analysis and even ergodic optimization for flows in general (see section 3).

We also attacked the full problem of equivalence with respect to continuous flows in general, and it turns out that this problem is intrinsically connected to the following dynamical embedding problem, which is also stated as an open question in [BHVZ21]:

Given a continuous flow Φ on a metric space X and a continuous function $\widetilde{b} : X \to \mathbb{R}$, is there a continuous function $b : X \to \mathbb{R}$ such that

$$\widetilde{b} = \int_0^1 (b \circ \phi_s) ds \quad ?$$

We stress here that a positive answer for this embedding problem implies a positive answer for the general physical equivalence problem.

In this work we are able to describe a sufficient condition on the function and on the flow where the dynamical embedding problem can be answered affirmatively. On the other hand, we also give a simple counter-example, showing that the embedding problem cannot be solved in full generality (see section 2.5). Overall, concerning the general physical equivalence problem, we give a positive answer in the case of continuous flows with uniquely ergodic time-one maps. However, as far as we know, the equivalence problem in full generality remains open.

The paper is organized as follows. We start proving our main equivalence result for suspension flows and, consequently, for hyperbolic flows. In the following, we also give natural sources of almost and asymptotically additive families of continuous functions. After that, we proceed to show how to extend the equivalence theorem to a class of expansive flows using the very recent work [GS22]. We finish section 2 with the general physical equivalence problem for continuous flows. In the final section, we conclude with some applications and consequences of the equivalence theorem, extending parts of the nonadditive thermodynamic formalism and multifractal analysis for continuous-time dynamical systems.

2 Main results

In this section, we are going to show that additive and asymptotically additive families of continuous functions are physically equivalent with respect to suspension flows and also with respect to some expansive flows. Moreover, we are going to give some natural examples of asymptotically additive families and also explore the general problem of physical equivalence for continuous flows.

2.1 Suspension flows

Let X be a compact metric space, $T\colon X \to X$ a homeomorphism and $\tau\colon X \to (0, \infty)$ a positive continuous function. Consider the space

$$W = \big\{(x, s) \in X \times \mathbb{R} : 0 \leq s \leq \tau(x)\big\}$$

and let Y be the set obtained from W identifying $(x, \tau(x))$ with $(T(x), 0)$ for each $x \in X$. Then a certain distance introduced by Bowen and Walters in [BW72] makes Y a compact metric space. The *suspension flow* over T with height function τ is the flow $\Phi = (\phi_t)_{t\in\mathbb{R}}$ on Y with the maps $\phi_t\colon Y \to Y$ defined by $\phi_t(x,s) = (x, s+t)$.

Let μ be a T-invariant probability measure on X. One can show that μ induces a Φ-invariant probability measure ν on Y such that

$$\int_\Lambda g\,d\nu = \frac{\int_X I_g\,d\mu}{\int_X \tau\,d\mu}$$

for any continuous function $g\colon Y \to \mathbb{R}$, where $I_g(x) = \int_0^{\tau(x)} (g\circ\phi_s)(x)\,ds$. Conversely, any Φ-invariant probability measure ν on Y is of this form for some T-invariant probability measure μ on X. Abramov's entropy formula says that

$$h_\nu(\Phi) = \frac{h_\mu(T)}{\int_X \tau\,d\mu}.$$

The next result establishes the physical equivalence between asymptotically additive and additive families of functions with respect to suspension flows.

Theorem 1. *Let $\Phi = (\phi_t)_{t\in\mathbb{R}}$ be a suspension flow over a continuous map $T : X \to X$ and $a = (a_t)_{t\geq 0}$ be an asymptotically additive family of continuous functions with respect to Φ. Then, there exists a continuous function $b : Y \to \mathbb{R}$ such that*

$$\lim_{t\to\infty} \frac{1}{t}\left\| a_t - \int_0^t (b\circ\phi_s)ds \right\|_\infty = 0.$$

Consider the sequence of continuous functions $c = (c_n)_{n\geq 1}$ on X given by $c_n(x) = a_{\tau_n(x)}(x)$, where

$$\tau_n(x) = \sum_{k=0}^{n-1} \tau(T^k(x)) \quad \text{for every } x \in X.$$

Lemma 1. *There exists a continuous function $\xi : X \to \mathbb{R}$ such that*

$$\lim_{n\to\infty} \frac{1}{n} \sup_{x\in X} \left| c_n(x) - \sum_{k=0}^{n-1} \xi(T^k(x)) \right| = 0. \tag{1}$$

Proof. Since a is asymptotically additive with respect to the flow Φ, for each $\varepsilon > 0$ there exists a continuous function $b_\varepsilon : Y \to \mathbb{R}$ such that

$$\limsup_{t\to\infty} \frac{1}{t}\left\| a_t - \int_0^t (b_\varepsilon\circ\phi_s)ds \right\|_\infty \leq \varepsilon. \tag{2}$$

It follows from the proof of Lemma 15 in [BH21b] that

$$\int_0^{\tau_n(x)} (b_\varepsilon\circ\phi_s)(x)ds = \sum_{k=0}^{n-1} (I_{b_\varepsilon}\circ T^k)(x) \quad \text{for every } x \in Y \text{ and } n \geq 1.$$

Notice that for each $t>0$ there exists a unique $n\in\mathbb{N}$ such that $\tau_n(x)\leq t\leq\tau_{n+1}(x)$ with $t-\tau_n(x)\in[0,\sup\tau)$. Then, in particular from (2), we have

$$\begin{aligned}
&\limsup_{n\to\infty}\frac{1}{n}\sup_{x\in X}\left|c_n(x)-\sum_{k=0}^{n-1}(I_{b_\varepsilon}\circ T^k)(x)\right| \\
&=\limsup_{n\to\infty}\frac{1}{\tau_n(x)}\left(\frac{\tau_n(x)}{n}\right)\sup_{x\in X}\left|a_{\tau_n(x)}(x)-\int_0^{\tau_n(x)}(b_\varepsilon\circ\phi_s)(x)ds\right|\leq\varepsilon\sup\tau
\end{aligned}\tag{3}$$

for any $x\in X$. Since $\varepsilon>0$ is arbitrarily small, this implies that the sequence c is asymptotically additive with respect to the map $T:X\to X$. Now we can apply Theorem 1.2 in [Cun20] to guarantee the existence of a continuous function $\xi:X\to\mathbb{R}$ satisfying (1), as desired. □

Lemma 2. *There exists a continuous function $b:Y\to\mathbb{R}$ such that $I_b|_X=\xi$.*

Proof. Following [BRW04], we can just define $b:Y\to\mathbb{R}$ as

$$b(\phi_s(x))=\frac{\xi(x)}{\tau(x)}\psi'\left(\frac{s}{\tau(x)}\right)$$

for each $x\in X$ and $s\in[0,\tau(x)]$, where $\psi:[0,1]\to[0,1]$ is any nondecreasing C^1 function such that $\psi(0)=0$, $\psi(1)=1$ and $\psi'(0)=\psi'(1)=0$. □

Lemma 3. *We have*

$$\limsup_{t\to\infty}\frac{1}{t}\sup_{x\in X}|a_t(x)-a_{\tau_n(x)}(x)|=0.$$

Proof. Since a is asymptotically aditive, for each $\varepsilon>0$ there exists $b_\varepsilon:Y\to\mathbb{R}$ such that

$$\limsup_{t\to\infty}\frac{1}{t}\left\|a_t-\int_0^t(b_\varepsilon\circ\phi_s)ds\right\|_\infty\leq\varepsilon.$$

Moreover, following as in the proof of Lemma 15 in [BH21b], one can check that for each $t>0$ there exists a unique $n\in\mathbb{N}$ with $t=\tau_n(x)+\kappa$ for some $\kappa\in[0,\sup\tau]$ such that

$$\left|\int_0^t(b_\varepsilon\circ\phi_s)(x)ds-\sum_{k=0}^{n-1}(I_{b_\varepsilon}\circ T^k)(x)\right|\leq\|b_\varepsilon\|_\infty\sup\tau.$$

Then, since

$$\begin{aligned}
\sup_{x\in X}|a_t(x)-a_{\tau_n(x)}(x)|\leq&\sup_{x\in X}\left|a_t(x)-\int_0^t(b_\varepsilon\circ\phi_s)(x)ds\right|\\
&+\sup_{x\in X}\left|\int_0^t(b_\varepsilon\circ\phi_s)(x)ds-\sum_{k=0}^{n-1}(I_{b_\varepsilon}\circ T^k)(x)\right|\\
&+\sup_{x\in X}\left|c_n(x)-\sum_{k=0}^{n-1}(I_{b_\varepsilon}\circ T^k)(x)\right|
\end{aligned}$$

for every $t \geq 0$, it follows from (3) that

$$\limsup_{t\to\infty} \frac{1}{t} \sup_{x\in X} |a_t(x) - a_{\tau_n(x)}(x)| \leq \varepsilon(1+\sup\tau) + \limsup_{t\to\infty} \frac{1}{t}(\|b_\varepsilon\|_\infty \sup\tau) = \varepsilon(1+\sup\tau).$$

Hence, letting $\varepsilon \to 0$ we obtain

$$\limsup_{t\to\infty} \frac{1}{t} \sup_{x\in X} |a_t(x) - a_{\tau_n(x)}(x)| = 0,$$

as desired. □

Proof of Theorem 1. Using Lemma 2, let $b : Y \to \mathbb{R}$ be a function such that $I_b|_X = \xi$ and define the family of continuous functions $\Delta := (\Delta_t)_{t\geq 0}$ as

$$\Delta_t(x) := a_t(x) - \int_0^t (b\circ\phi_s)(x)ds \quad \text{for all } x \in Y \text{ and } t \geq 0. \tag{4}$$

For each $x \in X$, we have

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{t} \sup_{x\in X} |\Delta_t(x)| \leq \limsup_{t\to\infty} \frac{1}{t} \sup_{x\in X} \left| a_{\tau_n(x)}(x) - \int_0^{\tau_n(x)} (b\circ\phi_s)(x)ds \right| \\
&+ \limsup_{t\to\infty} \frac{1}{t} \sup_{x\in X} |a_t(x) - a_{\tau_n(x)}(x)| \\
&+ \limsup_{t\to\infty} \frac{1}{t} \sup_{x\in X} \left| \int_0^{\tau_n(x)} (b\circ\phi_s)(x)ds - \int_0^t (b\circ\phi_s)(x)ds \right| \\
&\leq \left(\frac{1}{\inf\tau}\right) \limsup_{n\to\infty} \frac{1}{n} \sup_{x\in X} \left| c_n(x) - \sum_{k=0}^{n-1} \xi(T^k(x)) \right| \\
&+ \limsup_{t\to\infty} \frac{1}{t} \sup_{x\in X} (|a_t(x) - a_{\tau_n(x)}(x)| + \|b\|_\infty \sup\tau).
\end{aligned}$$

Hence, it follows from the lemmas 1 and 3 that

$$\limsup_{t\to\infty} \frac{1}{t} \sup_{x\in X} |\Delta_t(x)| = 0. \tag{5}$$

We claim that

$$\limsup_{t\to\infty} \frac{1}{t} \sup_{s\in[0,\tau]} \sup_{x\in X} |\Delta_t(\phi_s(x))| = 0.$$

By the asymptotic additivity of a, given $\varepsilon > 0$ there exists $b_\varepsilon : Y \to \mathbb{R}$ such that, in particular,

$$\limsup_{t\to\infty} \frac{1}{t} \sup_{x\in X} \left| a_t(x) - \int_0^t (b_\varepsilon\circ\phi_u)(x)du \right| \leq \varepsilon \quad \text{and} \tag{6}$$

$$\limsup_{t\to\infty} \frac{1}{t-s} \sup_{x\in X} \left| a_{t-s}(\phi_s(x)) - \int_0^{t-s} (b_\varepsilon \circ \phi_{u+s})(x)du \right| \le \varepsilon \quad \text{for every } s \in [0, \sup \tau]. \quad (7)$$

Since

$$\begin{aligned}
&\sup_{x\in X} |a_{t-s}(\phi_s(x)) - a_t(x)| \le \sup_{x\in X} \left| a_{t-s}(\phi_s(x)) - \int_0^{t-s} (b_\varepsilon \circ \phi_{u+s})(x)du \right| \\
&+ \sup_{x\in X} \left| a_t(x) - \int_0^t (b_\varepsilon \circ \phi_u)(x)du \right| + \sup_{x\in X} \left| \int_s^t (b_\varepsilon \circ \phi_u)(x)du - \int_0^t (b_\varepsilon \circ \phi_u)(x)du \right| \\
&\le \sup_{x\in X} \left| a_{t-s}(\phi_s(x)) - \int_0^{t-s} (b_\varepsilon \circ \phi_{u+s})(x)du \right| + \sup_{x\in X} \left| a_t(x) - \int_0^t (b_\varepsilon \circ \phi_u)(x)du \right| \\
&+ \|b_\varepsilon\|_\infty \sup \tau
\end{aligned}$$

for every $t \ge s$ and $s \in [0, \sup \tau]$, it follows from (6) and (7) that

$$\frac{1}{t} \sup_{s\in[0,\tau]} \sup_{x\in X} |a_{t-s}(\phi_s(x)) - a_t(x)| \le 2\varepsilon + 2\varepsilon + \varepsilon = 5\varepsilon \quad \text{for } t \text{ sufficiently large.}$$

Then, the arbitrariness of ε gives that

$$\limsup_{t\to\infty} \frac{1}{t} \sup_{s\in[0,\tau]} \sup_{x\in X} |a_{t-s}(\phi_s(x)) - a_t(x)| = 0. \quad (8)$$

From (4) one can check that

$$\sup_{s\in[0,\tau]} \sup_{x\in X} |\Delta_{t-s}(\phi_s(x)) - \Delta_t(x)| \le \sup_{s\in[0,\tau]} \sup_{x\in X} |a_{t-s}(\phi_s(x)) - a_t(x)| + \|b\|_\infty \sup \tau$$

for every $t \ge s$. Hence, it follows from (8) that

$$\limsup_{t\to\infty} \frac{1}{t} \sup_{s\in[0,\tau]} \sup_{x\in X} |\Delta_{t-s}(\phi_s(x)) - \Delta_t(x)| = 0. \quad (9)$$

Now observing that

$$\sup_{s\in[0,\tau]} \sup_{x\in X} |\Delta_{t-s}(\phi_s(x))| \le \sup_{s\in[0,\tau]} \sup_{x\in X} |\Delta_{t-s}(\phi_s(x)) - \Delta_t(x)| + \sup_{x\in X} |\Delta_t(x)|$$

for every $t \ge s$, from (5) and (9) we get

$$\limsup_{t\to\infty} \frac{1}{t} \sup_{s\in[0,\tau]} \sup_{x\in X} |\Delta_t(\phi_s(x))| = 0,$$

and the claim is proved. $\square$

Now for each $y \in Y$ there exist $x \in X$ and $s \in [0, \sup \tau]$ such that $y = \phi_s(x)$. Then

$$|\Delta_t(y)| = |\Delta_t(\phi_s(x))| \leq \sup_{s\in[0,\tau]} \sup_{x\in X} |\Delta_t(\phi_s(x))| \quad \text{for each } t \geq 0,$$

which readily implies that

$$\limsup_{t\to\infty} \frac{1}{t} \|\Delta_t\|_\infty \leq \limsup_{t\to\infty} \frac{1}{t} \sup_{s\in[0,\tau]} \sup_{x\in X} |\Delta_t(\phi_s(x))| = 0.$$

Since, in particular, Δ_n is asymptotically aditive with respect to the map ϕ_1 on Y, Lemma A.3 in [FH10] guarantees that the limit $\lim_{t\to\infty} \frac{1}{t}\|\Delta_t\|_\infty$ exists. Therefore

$$\lim_{t\to\infty} \frac{1}{t} \|\Delta_t\|_\infty = 0,$$

and the theorem is proved. □

Definition 1. Let X and Y be topological spaces and consider the flows Φ on X and Ψ on Y. We say that (X, Φ) is *topologically conjugate* to (Y, Ψ) if there exists an homeomorphism $h : X \to Y$ such that $(h \circ \phi_t)(x) = (\psi_t \circ h)(x)$ for every $t \in \mathbb{R}$ and every $x \in X$. Moreover, we say that (X, Φ) is C^r*-conjugate* to (Y, Ψ) if h is C^r.

Corollary 2. *Let $\Phi = (\phi_t)_{t\in\mathbb{R}}$ be a continuous flow on a compact metric space M. Suppose that Φ is topologically conjugate to a suspension flow. Then, for every asymptotically additive family of continuous functions $a = (a_t)_{t\in\mathbb{R}}$ there exists a continuous function $b : M \to \mathbb{R}$ such that*

$$\lim_{t\to\infty} \frac{1}{t} \left\| a_t - \int_0^t (b \circ \phi_s) ds \right\|_\infty = 0.$$

Proof. Let Ψ be a suspension flow on a compact metric space N and $h : M \to N$ the homeomorphism conjugating (M, Φ) and (N, Ψ). Suppose that $a = (a_t)_{t\geq 0}$ is an asymptotically additive family of continuous functions with respect to Φ. One can easily check that the family $(a_t \circ h^{-1})_{t\geq 0}$ is asymptotically additive with respect to Ψ. By Theorem 1, there exists a continuous function $\widetilde{b} : N \to \mathbb{R}$ such that

$$\lim_{t\to\infty} \frac{1}{t} \sup_{y\in N} \left| (a_t \circ h^{-1})(y) - \int_0^t (\widetilde{b} \circ \psi_s)(y) ds \right| = 0.$$

Since for each $y \in N$ there exists a unique $x \in M$ such that $h(x) = y$, we have

$$\left| a_t(h^{-1}(y)) - \int_0^t (\widetilde{b} \circ \psi_s)(y) ds \right| = \left| a_t(x) - \int_0^t ((\widetilde{b} \circ h) \circ \phi_s)(x) ds \right|.$$

Hence

$$\lim_{t\to\infty} \frac{1}{t} \sup_{x\in M} \left| a_t(x) - \int_0^t (b \circ \phi_s)(x) ds \right| = 0,$$

where $b := \widetilde{b} \circ h : M \to \mathbb{R}$.

□

Example 1. Let $\mathbb{T}^n := \mathbb{R}/\mathbb{Z} \times \cdots \times \mathbb{R}/\mathbb{Z}$ be the n-torus. Letting $\alpha \in \mathbb{R}^n$, the linear flow $\Phi = (\phi_t)_{t\in\mathbb{R}}$ on $\mathbb{T}^n$ in the direction α is defined by $\phi_t(x) = x + t\alpha \mod 1$.

One can see that every linear flow on the n-torus is C^∞- conjugate to a suspension flow (see for example [FH20]). Then, it follows from Corollary 2 that asymptotically additive and additive families of continuous functions are physically equivalent with respect to the flow Φ. □

2.2 Hyperbolic flows and Markov partitions

Let $\Phi = (\phi_t)_{t\in\mathbb{R}}$ be a C^1 flow on a smooth manifold M. A compact Φ-invariant set $\Lambda \subset M$ is called a *hyperbolic set for* Φ if there exists a splitting

$$T_\Lambda M = E^s \oplus E^u \oplus E^0$$

and constants $c > 0$ and $\lambda \in (0, 1)$ such that for each $x \in \Lambda$:

1. the vector $(d/dt)\phi_t(x)|_{t=0}$ generates $E^0(x)$;
2. for each $t \in \mathbb{R}$ we have
$$d_x\phi_t E^s(x) = E^s(\phi_t(x)) \quad \text{and} \quad d_x\phi_t E^u(x) = E^u(\phi_t(x));$$
3. $\|d_x\phi_t v\| \leq c\lambda^t\|v\|$ for $v \in E^s(x)$ and $t > 0$;
4. $\|d_x\phi_{-t} v\| \leq c\lambda^t\|v\|$ for $v \in E^u(x)$ and $t > 0$.

Given a hyperbolic set Λ for a flow Φ, for each $x \in \Lambda$ and any sufficiently small $\varepsilon > 0$ we define

$$A^s(x) = \big\{y \in B(x, \varepsilon) : d(\phi_t(y), \phi_t(x)) \searrow 0 \text{ when } t \to +\infty\big\}$$

and

$$A^u(x) = \big\{y \in B(x, \varepsilon) : d(\phi_t(y), \phi_t(x)) \searrow 0 \text{ when } t \to -\infty\big\}.$$

Moreover, let $V^s(x) \subset A^s(x)$ and $V^u(x) \subset A^u(x)$ be the largest connected components containing x. These are smooth manifolds, called respectively *(local) stable and unstable manifolds* of size ε at the point x, satisfying:

1. $T_xV^s(x) = E^s(x)$ and $T_xV^u(x) = E^u(x)$;
2. for each $t > 0$ we have
$$\phi_t(V^s(x)) \subset V^s(\phi_t(x)) \quad \text{and} \quad \phi_{-t}(V^u(x)) \subset V^u(\phi_{-t}(x));$$
3. there exist $\kappa > 0$ and $\mu \in (0, 1)$ such that for each $t > 0$ we have
$$d(\phi_t(y), \phi_t(x)) \leq \kappa\mu^t d(y, x) \quad \text{for } y \in V^s(x)$$
and
$$d(\phi_{-t}(y), \phi_{-t}(x)) \leq \kappa\mu^t d(y, x) \quad \text{for } y \in V^u(x).$$

We recall that a set Λ is said to be *locally maximal* (with respect to a flow Φ) if there exists an open neighborhood U of Λ such that

$$\Lambda = \bigcap_{t \in \mathbb{R}} \phi_t(U).$$

Given a locally maximal hyperbolic set Λ and a sufficiently small $\varepsilon > 0$, there exists $\delta > 0$ such that if $x, y \in \Lambda$ satisfy $d(x, y) \leq \delta$, then there exists a unique $t = t(x, y) \in [-\varepsilon, \varepsilon]$ such that

$$[x, y] := V^s(\phi_t(x)) \cap V^u(x)$$

is a single point in Λ.

Now let us recall the notion of Markov partitions for continuous-time dynamical systems. Consider an open smooth disk $D \subset M$ of dimension $\dim M - 1$ that is transverse to Φ and take $x \in D$. Let $U(x)$ be an open neighborhood of x diffeomorphic to $D \times (-\varepsilon, \varepsilon)$. Then the projection $\pi_D \colon U(x) \to D$ defined by $\pi_D(\phi_t(y)) = y$ is differentiable. We say that a closed set $R \subset \Lambda \cap D$ is a *rectangle* if $R = \overline{\operatorname{int} R}$ and $\pi_D([x, y]) \in R$ for $x, y \in R$.

Consider rectangles $R_1, \ldots, R_k \subset \Lambda$ (each contained in some open smooth disk transverse to the flow) such that

$$R_i \cap R_j = \partial R_i \cap \partial R_j \quad \text{for } i \neq j.$$

Let $Z = \bigcup_{i=1}^k R_i$. We assume that there exists $\varepsilon > 0$ such that:

1. $\Lambda = \bigcup_{t \in [0, \varepsilon]} \phi_t(Z)$;
2. whenever $i \neq j$, either

$$\phi_t(R_i) \cap R_j = \emptyset \quad \text{for all } t \in [0, \varepsilon]$$

or

$$\phi_t(R_j) \cap R_i = \emptyset \quad \text{for all } t \in [0, \varepsilon].$$

Now define the function $\tau \colon \Lambda \to \mathbb{R}_0^+$ by

$$\tau(x) = \min\{t > 0 : \phi_t(x) \in Z\},$$

and the map $T \colon \Lambda \to Z$ by

$$T(x) = \phi_{\tau(x)}(x). \tag{10}$$

The restriction T_Z of T to Z is invertible and we have $T^n(x) = \phi_{\tau_n(x)}(x)$, where

$$\tau_n(x) = \sum_{i=0}^{n-1} \tau(T^i(x)).$$

The collection $R_1, \ldots, R_k$ is said to be a *Markov partition* for Φ on Λ if

$$T(\operatorname{int}(V^s(x) \cap R_i)) \subset \operatorname{int}(V^s(T(x)) \cap R_j)$$

and

$$T^{-1}(\operatorname{int}(V^u(T(x)) \cap R_j)) \subset \operatorname{int}(V^u(x) \cap R_i)$$

for every $x \in \operatorname{int} T(R_i) \cap \operatorname{int} R_j$ and $i, j = 1, \ldots, k$. By work of Bowen [Bow73] and Ratner [Rat73], any locally maximal hyperbolic set Λ has Markov partitions of arbitrarily small diameter and the function τ is Hölder continuous on each domain of continuity.

Given a Markov partition $R_1, \ldots, R_k$ for a flow Φ on a locally maximal hyperbolic set Λ, we consider the $k \times k$ matrix A with entries

$$a_{ij} = \begin{cases} 1 & \text{if } \operatorname{int} T(R_i) \cap R_j \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

where T is the map in (10). We also consider the set

$$\Sigma_A = \left\{ (\cdots i_{-1} i_0 i_1 \cdots) : a_{i_n i_{n+1}} = 1 \text{ for } n \in \mathbb{Z} \right\} \subset \{1, \ldots, k\}^{\mathbb{Z}}$$

and the shift map $\sigma \colon \Sigma_A \to \Sigma_A$ defined by $\sigma(\cdots i_0 \cdots) = (\cdots j_0 \cdots)$, where $j_n = i_{n+1}$ for each $n \in \mathbb{Z}$. We denote by Σ_n the set of Σ_A-admissible sequences of length n, that is, the finite sequences $(i_1 \cdots i_n)$ for which there exists $(\cdots j_0 j_1 j_2 \cdots) \in \Sigma_A$ such that $(i_1 \ldots i_n) = (j_1 \cdots j_n)$. Finally, we define a *coding map* $\pi \colon \Sigma_A \to Z$ by

$$\pi(\cdots i_0 \cdots) = \bigcap_{n \in \mathbb{Z}} R_{i_{-n} \cdots i_n},$$

where $R_{i_{-n} \cdots i_n} = \bigcap_{l=-n}^{n} \overline{T_Z^{-l} \operatorname{int} R_{i_l}}$. The following properties hold:

1. $\pi \circ \sigma = T \circ \pi$;
2. π is Hölder continuous and onto;
3. π is one-to-one on a full measure set with respect to any ergodic measure of full support and on a residual set.

Given $\beta > 1$, we equip Σ_A with the distance d_β defined by

$$d_\beta(\omega, \omega') = \begin{cases} \beta^{-n} & \text{if } \omega \neq \omega', \\ 0 & \text{if } \omega = \omega', \end{cases}$$

where $n = n(\omega, \omega') \in \mathbb{N} \cup \{0\}$ is the smallest integer such that $i_n(\omega) \neq i_n(\omega')$ or $i_{-n}(\omega) \neq i_{-n}(\omega')$. One can always choose β so that $\tau \circ \pi$ is Lipschitz.

Now let ν be a T_Z-invariant probability measure on Z. One can show that ν induces a Φ-invariant probability measure μ on Λ such that

$$\int_\Lambda g\, d\mu = \frac{\int_Z \int_0^{\tau(x)} (g \circ \phi_s)(x)\, ds\, d\nu}{\int_Z \tau\, d\nu} \tag{11}$$

for any continuous function $g\colon \Lambda \to \mathbb{R}$. In fact, any Φ-invariant probability measure μ on Λ is of this form for some T_Z-invariant probability measure ν on Z. Abramov's entropy formula says that

$$h_\mu(\Phi) = \frac{h_\nu(T_Z)}{\int_Z \tau\, d\nu}. \tag{12}$$

By (11) and (12) we obtain

$$h_\mu(\Phi) + \int_\Lambda g\, d\mu = \frac{h_\nu(T_Z) + \int_Z I_g\, d\nu}{\int_Z \tau\, d\nu}, \tag{13}$$

where $I_g(x) = \int_0^{\tau(x)} (g \circ \phi_s) ds$. Since $\tau > 0$ on Z, it follows from (13) that

$$P_\Phi(g) = 0 \quad \text{if and only if} \quad P_{T_Z}(I_g) = 0,$$

where $P_\Phi(g)$ is the topological pressure of g with respect to Φ and $P_{T_Z}(I_g)$ is the topological pressure of I_g with respect to the map T_Z. When $P_\Phi(g) = 0$, this implies that μ is an equilibrium measure for g if and only if ν is an equilibrium measure for I_g.

As a direct consequence of the existence of Markov partitions for locally maximal hyperbolic sets together with Theorem 1, we obtain the following result:

Corollary 3. *Let Λ be a locally maximal hyperbolic set for a C^1 flow $\Phi = (\phi_t)_{t\in\mathbb{R}}$ and suppose that $a = (a_t)_{t\geq 0}$ is an asymptotically additive family of continuous functions with respect to Φ. Then, there exists a continuous function $b : \Lambda \to \mathbb{R}$ such that*

$$\lim_{t\to\infty} \frac{1}{t} \left\| a_t - \int_0^t (b \circ \phi_s) ds \right\|_\infty = 0.$$

2.3 Expansive flows

Let (X, d) be a metric space and $T : X \to X$ a dynamical system. T is said to be *expansive* if there exists $\delta > 0$ such that $d(T^n(x), T^n(y)) < \delta$ for all $n \in \mathbb{Z}$ implies $x = y$.

The definition for continuous-time dynamical systems is more refined. A flow Φ on X is said to be *expansive* if for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $d(\phi_t(x), \phi_{s(t)}(y)) < \delta$ for all $t \in \mathbb{R}$, for points x and y and a continuous map $s : \mathbb{R} \to \mathbb{R}$ with $s(0) = 0$, then there exists a time $|t| < \varepsilon$ such that $\phi_t(x) = y$ (see [BW72]).

Let X and Y be metric spaces and consider the flows Φ on X and Ψ on Y. Suppose that $\pi : Y \to X$ is a *topological extension* from (X, Φ) to (Y, Ψ), that is, a surjective map topologically conjugating (Φ, X) and (Ψ, Y). The extension $\pi : Y \to X$ is said to be *strongly isomorphic* if there exists a full set $E \subset X$ such that $\pi : \pi^{-1}(E) \subset Y \to X$ is one-to-one (see for example [Bur19]).

Advancing the main results in [Bur19], recently Gutman and Shi proved the following:

Theorem 4 ([GS22, Theorem B])**.** *Let X be a compact finite-dimensional space and Φ an expansive flow on X. Then, (X, Φ) is strongly isomorphic to a suspension flow over a subshift of finite type.*

This result together with Theorem 1 gives the following:

Theorem 5. *Let $\Phi = (\phi_t)_{t\in\mathbb{R}}$ be a continuous expansive flow on a compact finite-dimensional metric space M, and let $a = (a_t)_{t\geq 0}$ be an asymptotically additive family of continuous functions. Then, there exists a continuous function $b : M \to \mathbb{R}$ and a full set $N \subset M$ such that*

$$\lim_{t\to\infty} \frac{1}{t} \sup_{x\in N} \left| a_t(x) - \int_0^t (b \circ \phi_s)(x) ds \right| = 0. \tag{14}$$

Moreover, if Φ admits an invariant measure with full support then

$$\lim_{t\to\infty} \frac{1}{t} \left\| a_t - \int_0^t (b \circ \phi_s) ds \right\|_\infty = 0.$$

Proof. By Theorem 4, there exists a full set $N \subset X$ such that (N, Φ) and (Y, Ψ) are topologically conjugate, where Ψ is a suspension flow over a subshift of finite type. Then, (14) follows directly from Corollary 2.

Now suppose that $\nu \in \mathcal{M}(\Phi)$ is a measure with full support. Then, one can see that N is dense on the whole space M. Since the function

$$x \mapsto D_t(x) := \left| a_t(x) - \int_0^t (b \circ \phi_s)(x) ds \right|$$

is continuous for every $t \geq 0$, we have $\sup D_t(M) = \sup D_t(\overline{N}) \leq \sup \overline{D_t(N)} = \sup D_t(N)$ for every $t \geq 0$. This together with (14) yield

$$\lim_{t\to\infty} \frac{1}{t} \left\| a_t - \int_0^t (b \circ \phi_s) ds \right\|_\infty = \lim_{t\to\infty} \frac{1}{t} \sup_{x\in M} D_t(x) \leq \lim_{t\to\infty} \frac{1}{t} \sup_{x\in N} D_t(x) = 0,$$

and the theorem is proved. □

The following result is a weaker notion of equivalence in the case of expansive flows.

Corollary 6. *Let $\Phi = (\phi_t)_{t\in\mathbb{R}}$ be a continuous expansive flow on a compact finite-dimensional metric space M, and let $a = (a_t)_{t\geq 0}$ be an asymptotically additive family of continuous functions. Then, there exists a continuous function $b : M \to \mathbb{R}$ such that*

$$\lim_{t\to\infty} \frac{1}{t} \int_M a_t d\mu = \int_M b d\mu$$

for every measure $\mu \in \mathcal{M}(\Phi)$.

Proof. It follows directly from Theorem 5 and Birkhoff's ergodic theorem. □

Observe that volume preserving continuous expansive flows on compact finite-dimensional manifolds satisfy all the hypotheses of Theorem 5.

2.4 Examples: conformal and non-conformal hyperbolic flows

We will now introduce a source of asymptotically additive families of continuous potentials.

2.4.1 Conformal flows

We say that a C^1 flow Φ is *conformal* on a hyperbolic set Λ if there exist continuous functions $Q^s, Q^u \colon \Lambda \times \mathbb{R} \to \mathbb{R}$ such that

$$d_x\phi_t|E^s(x) = Q^s(x,t)J^s(x,t) \quad \text{and} \quad d_x\phi_t|E^u(x) = Q^u(x,t)J^u(x,t)$$

for every $x \in \Lambda$ and $t \in \mathbb{R}$, where

$$J^s(x,t) \colon E^s(x) \to E^s(\phi_t(x)) \quad \text{and} \quad J^u(x,t) \colon E^u(x) \to E^u(\phi_t(x))$$

are isometries. For example, if

$$\dim E^s(x) = \dim E^u(x) = 1 \quad \text{for } x \in \Lambda,$$

then the flow is conformal on Λ. Proceeding as in [PS01] we define:

$$\Xi_s(x) := \frac{\partial}{\partial t} \log |Q^s(x,t)|_{t=0} = \frac{\partial}{\partial t} \log \|d_x\phi_t|E^s(x)\|_{t=0} = \lim_{t\to 0} \frac{\log \|d_x\phi_t|E^s(x)\|}{t} \tag{15}$$

and

$$\Xi_u(x) := \frac{\partial}{\partial t} \log |Q^u(x,t)|_{t=0} = \frac{\partial}{\partial t} \log \|d_x\phi_t|E^u(x)\|_{t=0} = \lim_{t\to 0} \frac{\log \|d_x\phi_t|E^u(x)\|}{t}. \tag{16}$$

Since the flow Φ is of class C^1, using 2-norms one can write

$$\begin{aligned}\lim_{t\to 0} \frac{\log \|d_x\phi_t|E^s(x)\|}{t} &= \lim_{t\to 0} \frac{\log(\|d_x\phi_t|E^s(x)\|^2)}{2t} = \lim_{t\to 0} \frac{\langle d_x\phi_t|E^s(x), \frac{\partial}{\partial t}(d_x\phi_t|E^s(x))\rangle}{\|d_x\phi_t|E^u(x)\|^2} \\ &= \left\langle \mathrm{Id}|E^s(x), \frac{\partial}{\partial t}(d_x\phi_t|E^s(x))|_{t=0} \right\rangle\end{aligned}$$

and, similarly,

$$\lim_{t\to 0} \frac{\log \|d_x\phi_t|E^u(x)\|}{t} = \left\langle \mathrm{Id}|E^u(x), \frac{\partial}{\partial t}(d_x\phi_t|E^u(x))|_{t=0} \right\rangle.$$

In particular, the functions Ξ_s and Ξ_u are well defined. For an adapted norm $\|\cdot\|$ (that is, a norm for which one can take $c = 1$ in the definition of a hyperbolic set), we obtain

$$\Xi_s(x) = \lim_{t\to 0^+} \frac{\log \|d_x\phi_t|E^s(x)\|}{t} \le \log\lambda < 0$$

and

$$\Xi_u(x) = \lim_{t\to 0^+} \frac{\log \|d_x\phi_t|E^u(x)\|}{t} \ge -\log\lambda > 0$$

for all $x \in \Lambda$. Moreover, for every $x \in \Lambda$ and $t \in \mathbb{R}$, it follows from (15) and (16) that

$$\|d_x\phi_t v\| = \|v\| \exp\left(\int_0^t \Xi_s(\phi_\tau(x))\,d\tau\right) \quad \text{for } v \in E^s(x) \tag{17}$$

and

$$\|d_x\phi_t v\| = \|v\| \exp\left(\int_0^t \Xi_u(\phi_\tau(x))\,d\tau\right) \quad \text{for } v \in E^u(x). \tag{18}$$

In this case, notice that

$$(\log \|d_x\phi_t|E^s(x)\|)_{t\ge 0} \quad \text{and} \quad (\log \|d_x\phi_t|E^u(x)\|)_{t\ge 0}$$

are additive families of continuous functions with respect to Φ.

2.4.2 Non-conformal flows with bounded distortion

Let Λ be an hyperbolic set for a C^1 flow $\Phi = (\phi_t)_{t\in\mathbb{R}}$. Moreover, let $E^s(x)$ and $E^u(x)$ be the stable and unstable spaces at x. We say that Φ has *bounded distortion* (in the sense of [PS01]) if there exist constants $C_1 > 0$, $C_2 > 0$ and Hölder continuous functions $b^s, b^u \colon \Lambda \to \mathbb{R}$ such that

$$C_1\|v\| \exp \int_0^t (b^s \circ \phi_\tau)(x)\,d\tau \le \|d_x\phi_t v\| \le C_2\|v\| \exp \int_0^t (b^s \circ \phi_\tau)(x)\,d\tau$$

for $v \in E^s(x)$, and

$$C_1\|v\| \exp \int_0^t (b^u \circ \phi_\tau)(x)\,d\tau \le \|d_x\phi_t v\| \le C_2\|v\| \exp \int_0^t (b^u \circ \phi_\tau)(x)\,d\tau$$

for $v \in E^u(x)$. In this case one can easily verify that the families $a^s = (a^s_t)_{t\ge 0}$ and $a^u = (a^u_t)_{t\ge 0}$ given by

$$a^s_t(x) = \log \|d_x\phi_t|_{E^s(x)}\| \quad \text{and} \quad a^u_t(x) = \log \|d_x\phi_t|_{E^u(x)}\|$$

are almost additive with respect to Φ and satisfy

$$\lim_{t\to\infty} \frac{1}{t}\left\| a^s_t - \int_0^t (b^s \circ \phi_\tau)\,d\tau \right\|_\infty = 0$$

and

$$\lim_{t\to\infty} \frac{1}{t}\left\| a^u_t - \int_0^t (b^u \circ \phi_\tau)\,d\tau \right\|_\infty = 0.$$

2.4.3 Quasiconformal flows

Now let Λ be an hyperbolic set for a C^1 flow $\Phi = (\phi_t)_{t\in\mathbb{R}}$, and consider the functions $K^s, K^u : \Lambda \times \mathbb{R} \to \mathbb{R}$ given by

$$K^s(x,t) = \frac{\max\{\|d_x\phi_t v\| : v \in E^s(x), \|v\| = 1\}}{\min\{\|d_x\phi_t v\| : v \in E^s(x), \|v\| = 1\}} \tag{19}$$

and

$$K^u(x,t) = \frac{\max\{\|d_x\phi_t v\| : v \in E^u(x), \|v\| = 1\}}{\min\{\|d_x\phi_t v\| : v \in E^u(x), \|v\| = 1\}}. \tag{20}$$

We say that Φ is *uniformly quasiconformal* if the functions K^s and K^u are uniformly bounded for all $x \in \Lambda$ and $t \in \mathbb{R}$. Observe that when Φ is conformal on Λ, it follows directly from (17) and (18) that $K^s(x,t) = 1$ and $K^u(x,t) = 1$ for all $x \in \Lambda$ and all $t \in \mathbb{R}$. The notion of a uniformly quasiconformal hyperbolic map is analogous (see [Sad05]). Observe that conformal flows are quasiconformal and an Anosov diffeomorphism is quasiconformal if and only if its suspension flow is quasiconformal (see for example [Fan05]). It follows from (19) and (20) that

$$\lim_{t\to\infty} \frac{1}{t}\|K^s(x,t)\|_\infty = 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{1}{t}\|K^u(x,t)\|_\infty = 0,$$

which readily implies that $(K^s(x,t))_{t\geq 0}$ and $(K^u(x,t))_{t\geq 0}$ are asymptotically additive families with respect to Φ.

2.4.4 Average conformal flows

Inspired by previous work [BCH10], it was introduced in [WWCZ20] a type of non-conformal hyperbolic maps, which can also be seen as a generalization of quasiconformal maps. Let Λ be an hyperbolic set for a diffeomorphism $f : \Lambda \to \Lambda$. The set Λ is called an *average conformal* hyperbolic set for f if it admits exactly two unique Lyapunov exponents, one strictly positive and the other strictly negative.

If $\Phi = (\phi_t)_{t\in\mathbb{R}}$ is a suspension flow over an average conformal hyperbolic map, it follows from Lemma 2.3 in [WWCZ20] that

$$\lim_{t\to\infty} \frac{1}{t}\| \log \|d_x\phi_t|_{E^s(x)}\| - \log \|(d_x\phi_t|_{E^s(x)})^{-1}\|^{-1}\|_\infty = 0,$$
$$\lim_{t\to\infty} \frac{1}{t}\| \log \|d_x\phi_t|_{E^u(x)}\| - \log \|(d_x\phi_t|_{E^u(x)})^{-1}\|^{-1}\|_\infty = 0 \quad \text{and}$$

$$\|(d_x\phi_t|_{E^s(x)})^{-1}\|^{-1} \leq |\det(d_x\phi_t|_{E^s(x)})|^{\frac{1}{d_s}} \leq \|d_x\phi_t|_{E^s(x)}\|,$$
$$\|(d_x\phi_t|_{E^u(x)})^{-1}\|^{-1} \leq |\det(d_x\phi_t|_{E^u(x)})|^{\frac{1}{d_u}} \leq \|d_x\phi_t|_{E^u(x)}\|,$$

where $d_s := \dim E^s$ and $d^u := \dim E^u$. Since $(\det(d_x\phi_t|_{E^u(x)})_{t\geq 0}$ and $(\det(d_x\phi_t|_{E^s(x)})_{t\geq 0}$ are additive families with respect to Φ, one can see that $a^i := (\| \log \|d_x\phi_t|_{E^i(x)}\|)_{t\geq 0}$ and

$b^i := (\| \log \|d_x\phi_t|_{E^i(x)}\| - \log \|(d_x\phi_t|_{E^i(x)})^{-1}\|^{-1})_{t\geq 0}$ are asymptotically additive families of continuous functions with respect to Φ for $i \in \{u, s\}$.

2.5 The general physical equivalence problem

We recall that a family of functions $a = (a_t)_{t\geq 0}$ is said to have *bounded variation* if for every $\kappa > 0$ there exists $\varepsilon > 0$ such that

$$|a_t(x) - a_t(y)| < \kappa \quad \text{whenever } y \in B_t(x, \varepsilon).$$

Inspired by the examples in subsection 2.4 and by the work [Cun20], one can ask the following questions:

Question A: Given an asymptotically additive family of continuous functions $a = (a_t)_{t\geq 0}$ with respect to a continuous flow $\Phi = (\phi_t)_{t\in\mathbb{R}}$ on a compact metric space X, is there any continuous function $b\colon X \to \mathbb{R}$ such that

$$\lim_{t\to\infty} \frac{1}{t}\left\| a_t - \int_0^t (b \circ \phi_s) ds \right\|_\infty = 0 \ ? \tag{21}$$

Question B: When $\Phi|_\Lambda$ is a hyperbolic flow and the family a is almost additive with bounded variation, is there any continuous function $b\colon \Lambda \to \mathbb{R}$ where the additive family

$$S_t b := \int_0^t (b \circ \phi_s) ds$$

has bounded variation and also satisfy (21)?

As we showed, Theorem 1 answers the **Question A** in the case of suspension flows and, in particular, in the case of hyperbolic flows. Moreover, Theorem 5 gives a setup where we also can positively answer the **Question A** in the case of expansive flows.

We notice that the **Question B** is open even in the case of discrete time dynamical systems (see [Cun20]). In general, positive answers to these questions are very useful for some extensions of the thermodynamic formalism and multifractal analysis for flows (see section 3).

Let us now give some directions on how to approach this problem in general. Let $\Phi = (\phi_t)_{t\in\mathbb{R}}$ be a continuous flow and let $a = (a_t)_{t\geq 0}$ be an asymptotically additive family of continuous functions with respect to Φ. For any function $c : X \to \mathbb{R}$, we have

$$\int_0^n (c \circ \phi_s) ds = \sum_{k=0}^{n-1} (\widetilde{c} \circ \phi_1^k) \quad \text{for every } n \geq 1 \tag{22}$$

where $\widetilde{c} := \int_0^1 (c \circ \phi_s) ds$, and one can check that $(a_n)_{n\geq 1}$ is an asymptotically additive sequence with respect to the map ϕ_1. Then, by Theorem 1.2 in [Cun20], there exists a continuous function $\widetilde{b} : X \to \mathbb{R}$ such that

$$\lim_{n\to\infty} \frac{1}{n} \left\| a_n - \sum_{k=0}^{n-1} \widetilde{b} \circ \phi_1^k \right\|_\infty = 0.$$

If there exists a continuous function $b : X \to \mathbb{R}$ such that $\widetilde{b} = \int_0^1 (b \circ \phi_s) ds$, from (22) we readily obtain that

$$\lim_{n\to\infty} \frac{1}{t} \left\| a_t - \int_0^t (b \circ \phi_s) ds \right\|_\infty = 0.$$

Just for illustration, here it goes a simple example:

Example 2. Let $\Phi = (\phi_t)_{t\in\mathbb{R}}$ be the flow $\phi_t(x) = e^t x$ on $\mathbb{R}^+ := (0, \infty]$. If $b(x) = 1/2 + \log x$, then for $a(x) = \log x$, we have

$$\int_0^1 (a \circ \phi_s)(x) ds = b(x) \quad \text{for every } x \in \mathbb{R}^+.$$

□

A less (but still) simple example:

Example 3. Let $\Phi = (\phi_t)_{t\in\mathbb{R}}$ be the flow $\phi_t(x) = e^t x$ defined on $\mathbb{R}$. If $b : \mathbb{R} \to \mathbb{R}$ is given by

$$b(x) = p(x)\left(\frac{e^d - 1}{d}\right),$$

where $p : \mathbb{R} \to \mathbb{R}$ is an homogeneous polynomial of degree d, then

$$\int_0^1 (p \circ \phi_s)(x) ds = b(x) \quad \text{for every } x \in \mathbb{R}.$$

□

Following this approach, we can ask:

Question C. Given a continuous flow Φ and a continuous function $\widetilde{b} : X \to \mathbb{R}$, is there any continuous function $b : X \to \mathbb{R}$ satisfying

$$\widetilde{b}(x) = \int_0^1 (b \circ \phi_s)(x) ds \quad \text{for every } x \in X \ ?$$

Question D. If we cannot give a positive answer to the previous question in general, what kind of functions and flows satisfy it?

In order to extend the theory of ergodic optimization for flows, the **Question C** was also posted in [BHVZ21].

The following example indicates a negative answer to **Question C** in the case where the function $\widetilde{b} : X \to \mathbb{R}$ is bounded.

Example 4. Let X be a compact metric space and $\Phi = (\phi_t)_{t\in\mathbb{R}}$ a continuous flow on X. Let $x^* \in X$ be a non-fixed point of Φ and consider $\widetilde{b} := \mathbb{1}_{x^*} : X \to \mathbb{R}$ the characteristic function of the set $\{x^*\}$. Suppose there exists a bounded function $b : X \to \mathbb{R}$ such that

$$\int_0^1 (b \circ \phi_s)(x)ds = \widetilde{b}(x) \quad \text{for all } x \in X. \tag{23}$$

This implies that

$$\int_0^1 (b \circ \phi_s)(x^*)ds = \widetilde{b}(x^*) = 1 \quad \text{and} \quad \int_0^1 (b \circ \phi_s)(x)ds = \widetilde{b}(x) = 0 \quad \text{for every } x \neq x^*.$$

Given $\delta > 0$, consider the point $z := \phi_\delta(x^*)$. Since x^* is not a fixed point, $z \neq x^*$ and we have

$$0 = \int_0^1 (b \circ \phi_s)(z)ds = \int_\delta^{\delta+1} (b \circ \phi_s)(x^*)ds$$

Then

$$\begin{aligned}\int_0^1 (b \circ \phi_s)(x^*)ds &= \int_0^\delta (b \circ \phi_s)(x^*)ds + \int_\delta^{\delta+1} (b \circ \phi_s)(x^*)ds + \int_{\delta+1}^1 (b \circ \phi_s)(x^*)ds \\ &= \int_0^\delta (b \circ \phi_s)(x^*)ds + \int_{\delta+1}^1 (b \circ \phi_s)(x^*)ds,\end{aligned}$$

which readily implies that

$$\left| \int_0^1 (b \circ \phi_s)(x^*)ds \right| \leq 2\delta \|b\|_\infty.$$

Since δ is arbitrary and the function b is bounded, taking $\delta < 1/(2\|b\|_\infty)$ we obtain

$$1 = |\widetilde{b}(x^*)| = \left| \int_0^1 (b \circ \phi_s)(x^*)ds \right| < 1,$$

which is a contradiction. In particular, since X is compact, there is no continuous function $b : X \to \mathbb{R}$ satisfying (23). □

Proposition 7. *Let Φ be a continuous flow on a metric space X. Suppose that for a given continuous function $\widetilde{b} : X \to \mathbb{R}$ there exists a continuous function $b : X \to \mathbb{R}$ such that*

$$\widetilde{b}(x) = \int_0^1 (b \circ \phi_s)(x)ds \quad \textit{for every } x \in X.$$

Then, we have

$$\lim_{t\to 0} \frac{\widetilde{b}(\phi_t(x)) - \widetilde{b}(x)}{t} = b(\phi_1(x)) - b(x) \quad \textit{for every } x \in X. \tag{24}$$

Moreover,

$$\int_X \left(\lim_{t\to 0} \frac{\widetilde{b}(\phi_t(x)) - \widetilde{b}(x)}{t} \right) d\mu(x) = 0 \quad \textit{for every } \mu \in \mathcal{M}(\phi_1).$$

Proof. Notice that

$$\frac{\widetilde{b}\circ\phi_t-\widetilde{b}}{t}=\frac{1}{t}\left(\int_t^{t+1}(b\circ\phi_s)ds-\int_0^1(b\circ\phi_s)ds\right)\quad\text{for all } t>0. \tag{25}$$

One can check that the function

$$t\mapsto I(t):=\int_t^{t+1}(b\circ\phi_s)ds-\int_0^1(b\circ\phi_s)ds$$

is uniformly continuous on $[0,\infty)$, differentiable on $(0,\infty)$ and satisfies $\lim_{t\to0} I(t)=0$. Hence, by the L'Hôspital's rule, we obtain that

$$\lim_{t\to0}\frac{I(t)}{t}=\lim_{t\to0}(b\circ\phi_{t+1}-b\circ\phi_t)=b\circ\phi_1-b.$$

This together with (25) proves the result. □

Remark. One can check directly that the functions in Example 2 and Example 3 satisfy the condition (24).

Observe that if there exists a constant $\beta\in\mathbb{R}$ such that

$$\lim_{t\to0}\frac{\widetilde{b}(\phi_t(x))-\widetilde{b}(x)}{t}=\beta\quad\text{for every } x\in X,$$

then Proposition 7 says that $\beta=0$. This fact is an inspiration for our *negative* answer to **Question C**:

Example 5. (Counter-example). Let $\Phi=(\phi_t)_{t\in\mathbb{R}}$ be the linear flow on $\mathbb{T}^2$ given by $\phi_t(x,y)=(x+t\alpha_1 \bmod 1, y+t\alpha_2 \bmod 1)$, with $0<\alpha_1+\alpha_2<1$. Let $\widetilde{b}:\mathbb{T}^2\to\mathbb{R}$ be the continuous function given by $\widetilde{b}(x,y)=(x+y) \bmod 1$. One can check that

$$\begin{aligned}\lim_{t\to0}\frac{\widetilde{b}(\phi_t(x,y))-\widetilde{b}(x,y)}{t}&=\lim_{t\to0}\frac{(x+y+t(\alpha_1+\alpha_2)) \bmod 1-(x+y) \bmod 1}{t}\\&=\lim_{t\to0}\frac{t(\alpha_1+\alpha_2) \bmod 1}{t}=\lim_{t\to0}\frac{t(\alpha_1+\alpha_2)}{t}=\alpha_1+\alpha_2\end{aligned}$$

for every $x\in\mathbb{T}^2$. If there exists a continuous function $b:\mathbb{T}^2\to\mathbb{R}$ such that $\widetilde{b}=\int_0^1(b\circ\phi_s)ds$, then Proposition 7 immediately gives that $\alpha_1+\alpha_2=0$, which is a contradiction. Therefore, there is no continuous function $b:\mathbb{T}^2\to\mathbb{R}$ such that

$$(x+y) \bmod 1=\int_0^1 b(\phi_s(x,y))ds.$$

□

Remark. Example 5 is a negative answer to the general embedding problem presented in [BHVZ21]. This counter-example can be naturally extended to the n-torus $\mathbb{T}^n$. Observe that even though we cannot guarantee in general a positive answer for **Question C** in the case of linear flows on $\mathbb{T}^n$, in Example 1 we were able to guarantee the physical equivalence between additive and asymptotically additive families of continuous potentials.

Notice that the converse of Proposition 7 is false. In fact, still considering Example 5, if we take $b(x, y) = (x + y) \bmod 1$, then $b(\phi_1(x, y)) - b(x, y) = (\alpha_1 + \alpha_2) \bmod 1 = \alpha_1 + \alpha_2$ for every $x \in \mathbb{T}^2$. But

$$\int_0^1 b(\phi_s(x, y))ds = \left[(x + y) \bmod 1 + \frac{\alpha_1 + \alpha_2}{2}\right] \bmod 1 \neq \widetilde{b}(x, y).$$

We will now define the notion of cohomology for flows and maps.

Definition 2. Let X be a metric space, Ψ a flow on X and $T : X \to X$ a map. We say that a function $g : X \to \mathbb{R}$ is Ψ-*cohomologous* to a function $h : X \to \mathbb{R}$ if there exists a bounded measurable function $q : X \to \mathbb{R}$ such that

$$g(x) - h(x) = \lim_{t \to 0} \frac{q(\psi_t(x)) - q(x)}{t} \quad \text{for every } x \in X.$$

Moreover, we say that g is T-*cohomologous* to h if there exists a bounded measurable function $r : X \to \mathbb{R}$ such that

$$g(x) - h(x) = r(T(x)) - r(x) \quad \text{for every } x \in X.$$

We also say that a function is a Ψ-*coboundary* (T-*coboundary*) when it is Ψ-cohomologous to the zero function (T-cohomologous to the zero function). The functions $q, r : X \to \mathbb{R}$ are usually called *transfer functions*.

The following result gives some directions for **Question D**:

Proposition 8. *Let $\Phi = (\phi_t)_{t \in \mathbb{R}}$ be a C^1 flow on a Riemmanian manifold X and $\widetilde{b} : X \to \mathbb{R}$ a C^1 ϕ_1-coboundary function admitting a C^1 transfer function. Then, there exists a continuous function $b : X \to \mathbb{R}$ such that*

$$\int_0^1 (b \circ \phi_s)ds = \widetilde{b}.$$

Moreover, the function b is a Φ-coboundary also admitting a C^1 transfer function.

Proof. Suppose that

$$\widetilde{b} = g \circ \phi_1 - g \quad \text{for some } C^1 \text{ function } g : X \to \mathbb{R}.$$

Defining a function $b : X \to \mathbb{R}$ as

$$b(x) = d_x g(x)\left(\frac{d}{ds}\phi_s(x)|_{s=0}\right) = \lim_{s\to 0}\frac{g(\phi_s(x)) - g(x)}{s},$$

we can check that b is continuous and

$$\int_0^t (b \circ \phi_s) ds = g \circ \phi_t - g \quad \text{for every } t \geq 0.$$

In particular, we have

$$\int_0^1 (b \circ \phi_s) ds = g \circ \phi_1 - g = \widetilde{b}.$$

Notice that $\widetilde{b}$ is a ϕ_1-coboundary and b is a Φ-coboundary, both admitting the same C^1 transfer function g. □

Remark. In Proposition 8, if X is a compact manifold, ϕ_1 is a C^∞ diffeomorphism and $\widetilde{b}$ is C^∞ then one can always guarantee the existence of a C^∞ function g such that $\widetilde{b} = g \circ \phi_1 - g$ (see [LMM86]).

A map T is said to be *uniquely ergodic* if it admits a unique T-invariant measure. In the same manner, we say that a flow Ψ is *uniquely ergodic* when there exists a unique Ψ-invariant measure.

Proposition 9. *Let $\Phi = (\phi_t)_{t\in\mathbb{R}}$ be a continuous flow on a compact metric space M and such that $\phi_1 : M \to M$ is uniquely ergodic. Then*

1. *$(a_t)_{t\geq 0}$ is an asymptotically additive family with respect to Φ if and only if $(a_n)_{n\geq 1}$ is an asymptotically additive sequence with respect to ϕ_1;*
2. *for each asymptotically additive family of continuous functions $a = (a_t)_{t\geq 0}$ there exists a continuos function $c : M \to \mathbb{R}$ such that*

$$\lim_{t\to+\infty}\frac{1}{t}\left\|a_t - \int_0^t (c \circ \phi_s) ds\right\|_\infty = 0.$$

Proof. Consider the linear operator $\mathcal{L} : C(X) \to C(X)$ given by

$$\mathcal{L}(a)(x) = \int_0^1 (a \circ \phi_s)(x) ds.$$

Then, one can see that $\|\mathcal{L}\| \leq 1$, where we are considering the norm

$$\|\mathcal{L}\| := \sup_{f\in C(X), \|f\|_\infty \neq 0}\frac{\|\mathcal{L}f\|_\infty}{\|f\|_\infty}.$$

Since $\mathcal{L}$ is a bounded linear operator, its spectrum $\sigma(\mathcal{L})$ is compact and is such that

$$\sigma(\mathcal{L}) \subset [-\|\mathcal{L}\|, \|\mathcal{L}\|] \subset [-1, 1].$$

In particular, this implies that there exists $\lambda > 1$ such that the linear operator $\mathcal{L} - \lambda I : C(X) \to C(X)$ is a bijection. This readily implies that given a continuous function $b : X \to \mathbb{R}$ there exists a unique continuous function $a : X \to \mathbb{R}$ such that

$$\int_0^1 (a \circ \phi_s) ds - \lambda a = b. \tag{26}$$

Now let ν be the unique ϕ_1-invariant measure and consider the function

$$c := a - \lambda \int_M a d\nu.$$

It follows from (26) that

$$\left\| a_n - \int_0^n (c \circ \phi_s) ds \right\|_\infty \leq \left\| a_n - \sum_{k=0}^{n-1} b \circ \phi_1^k \right\|_\infty + \left\| n\lambda \left(\int_M a d\nu - \sum_{k=0}^{n-1} a \circ \phi_1^k \right) \right\|_\infty. \tag{27}$$

Since ϕ_1 is uniquely ergodic, we have in particular that

$$\lim_{n\to\infty} \left\| \lambda \left(\int_M a d\nu - \frac{1}{n} \sum_{k=0}^{n-1} a \circ \phi_1^k \right) \right\|_\infty = 0$$

for every $a \in C(X)$.

Let us start proving item 1. By the arbitrariness of $b \in C(X)$, if $(a_n)_{n\geq 1}$ is asymptotically additive with respect to ϕ_1 then it follows directly from (27) that $(a_t)_{t\geq 0}$ is asymptotically additive with respect to Φ. The converse is immediate.

In order to prove item 2, notice that Theorem 1.2 in [Cun20] guarantees the existence of a function $b \in C(X)$ such that

$$\lim_{n\to\infty} \frac{1}{n} \left\| a_n - \sum_{k=0}^{n-1} b \circ \phi_1^k \right\|_\infty = 0.$$

Therefore, it readily follows from (27) that there exists a function $c \in C(X)$ such that

$$\lim_{t\to\infty} \frac{1}{t} \left\| a_t - \int_0^t (c \circ \phi_s) ds \right\|_\infty = 0,$$

as desired. □

Remark. Notice that if the time-one map ϕ_1 of a flow Φ is uniquely ergodic then the flow Φ itself is uniquely ergodic. From this, we can see that the setup of Proposition 9 is quite restrictive. We still don't know if the result remains true in full generality, without the unique ergodicity hypotheses.

3 Applications and consequences of Theorem 1

In this section, using Theorem 1, we are going to show how to extend some results of the nonadditive thermodynamic formalism and multifractal analysis for flows.

3.1 Nonadditive thermodynamic formalism

Let M be a compact metric space and $\Lambda \subset M$ be a locally maximal hyperbolic set for a topologically mixing C^1 flow Φ. Suppose that $a = (a_t)_{t\geq 0}$ is an asymptotically additive family of continiuous functions with tempered variation. Hence, Theorem 1 guarantees the existence of a continuous function $b : \Lambda \to \mathbb{R}$ such that

$$\sup_{\mu\in\mathcal{M}(\Phi)} \left\{ h_\mu(\Phi) + \lim_{t\to\infty} \frac{1}{t} \int_\Lambda a_t d\mu \right\} = \sup_{\mu\in\mathcal{M}(\Phi)} \left\{ h_\mu(\Phi) + \int_\Lambda b d\mu \right\}.$$

Moreover, by the definition of the nonadditive topological pressure introduced in [BH20], $P(a) = P\big((S_t b)_{t\geq 0}\big)$. Therefore, the classical variational principle for continuous flows implies that

$$P(a) = \sup_{\mu\in\mathcal{M}(\Phi)} \left\{ h_\mu(\Phi) + \lim_{t\to\infty} \frac{1}{t} \int_\Lambda a_t d\mu \right\}.$$

This is a variational principle for asymptotically additive families of continuous functions with respect to a hyperbolic flow. Notice that by Theorem 5 the variational principle is also valid for expansive continiuous flows admitting an invariant measure of full support.

In the case of locally maximal hyperbolic sets for flows or suspension flows in general, and expansive flows with measures of full support, this result extends Theorem 9 in [BH20] and Theorem 1.1 in [BH21a].

Based on this variational principle, we can also define the notion of equilibrium measures for asymptotically additive families of potentials. We say that $\nu \in \mathcal{M}(\Phi)$ is an equilibrium measure for a with respect to Φ if

$$P(a) = h_\nu(\Phi) + \lim_{t\to+\infty} \frac{1}{t} \int_\Lambda a_t d\nu.$$

Corollary 10. *Let $\Lambda \subset M$ be a locally maximal hyperbolic set for a C^1 flow Φ or suppose Φ is an expansive continuous flow admitting an invariant measure of full support. If a is an asymptotically additive family of functions with respect to Φ, then*

1. *the set of equilibrium measures for a is a non-empty compact and convex set;*
2. *each extreme point of the set of equilibrium measures is an ergodic measure.*

Proof. By Theorem 1 and Theorem 5, respectively for hyperbolic flows and expansive flows, the space of equilibrium measures for a is the same as the space of equilibrium measures for some continuous function. □

Theorem 3.5 in [BH21a] guarantees that for locally maximal hyperbolic sets for C^1 flows, every almost additive family with bounded variation admits a unique equilibrium measure. Since we don't know if **Question B** is true, we are not able to obtain this uniqueness result directly from the additive case (see [Fra77]).

3.2 Multifractal analysis: beyond families with unique equilibrium measures

Theorem 9 in [BH21a] establishes a conditional variational principle for almost additive families of potentials with uniqueness of equilibrium measures. In this section, using the work of Climenhaga in [Cli13], Cuneo in [Cun20], we are going to show how to obtain a conditional variational principle including more classes of almost additive families of functions, without using the uniqueness of equilibrium assumption. Moreover, as a direct consequence of Theorem 1, we also can extend the conditional variational principle to include asymptotically additive families of continuous potentials.

Let M be a compact metric space and $\Lambda \subset M$ be a locally maximal hyperbolic set for a C^1 topologically mixing flow $\Phi = (\phi_t)_{t\in\mathbb{R}}$. Given two asymptotically additive families of continuous functions $a = (a_t)_{t\geq 0}$ and $b = (b_t)_{t\geq 0}$, we consider the level sets

$$K_\alpha^\Phi(a,b) := \left\{ x \in \Lambda : \lim_{t\to+\infty} \frac{a_t(x)}{b_t(x)} = \alpha \right\}, \quad \alpha \in \mathbb{R}.$$

In this section, we consider $\mathcal{M}_{erg}(\Phi)$ as the set of ergodic Φ-invariant measures. We also use this same notation for maps.

Let $\mathcal{A}(M)$ be the set of all almost additive families of continuous functions and, respectively, $\mathcal{AA}(M)$ the set of asymptotically additive families of continuous functions $a = (a_t)_{t\geq 0}$ on M with tempered variation such that

$$\sup_{t\in[0,s]} \|a_t\|_\infty < \infty \quad \text{for some } s > 0,$$

and $\mathcal{E}(M) \subset \mathcal{A}(M)$ the subset of families having a unique equilibrium measure (the existence of almost additive families with unique equilibrium measures is guaranteed by Theorem 3.5 in [BH21a]).

Now let us define the notion of *u-dimension* for flows which was originally introduced in [BS00]. Let $\Phi = (\phi_t)_{t\in\mathbb{R}}$ be a continuous flow on a compact metric space X. Given a positive continuous function $u\colon X \to \mathbb{R}$, we consider the additive family of continuous functions $(u_t)_{t\geq 0}$ defined by

$$u_t(x) = \int_0^t (u \circ \phi_s)(x) ds$$

for every $x \in X$ and $t > 0$. For each $Z \subset X$ and $\alpha \in \mathbb{R}$, let

$$N(Z, u, \alpha, \varepsilon) = \lim_{T\to\infty} \inf_{\Gamma} \sum_{(x,t)\in\Gamma} e^{-\alpha u(x,t,\varepsilon)},$$

with the infimum taken over all countable sets $\Gamma \subset X \times [T, +\infty)$ covering Z. Finally, we define

$$\dim_{u,\varepsilon} Z = \inf\big\{\alpha \in \mathbb{R} : N(Z, u, \alpha, \varepsilon) = 0\big\}.$$

The limit

$$\dim_u Z := \lim_{\varepsilon \to 0} \dim_{u,\varepsilon} Z$$

exists and is called the u-dimension of the set Z with respect to the flow Φ. When $u \equiv 1$ the number $\dim_u Z$ coincides with the topological entropy $h(\Phi|_Z)$ of Φ on the set Z.

Theorem 11. *Let $\Lambda \subset M$ be a locally maximal hyperbolic set for a C^1 topologically mixing flow $\Phi = (\phi_t)_{t\in\mathbb{R}}$ such that $h(\Phi) < \infty$. Let $a = (a_t)_{t\geq 0}$ and $b = (b_t)_{t\geq 0}$ be asymptotically additive families of continuous functions in $\mathcal{AA}(\Lambda)$ and such that*

$$\lim_{t\to\infty} \frac{1}{t} \int_\Lambda b_t d\mu \geq 0 \quad \textit{for all } \mu \in \mathcal{M}(\Phi)$$

with equality only permitted when

$$\lim_{t\to\infty} \frac{1}{t} \int_\Lambda a_t d\mu \neq 0.$$

If $\alpha \notin \mathcal{P}(\mathcal{M}(\Phi))$, then $K_\alpha^\Phi(a,b) = \emptyset$. Moreover, if $\alpha \in \operatorname{int} \mathcal{P}(\mathcal{M}(\Phi))$ then $K_\alpha^\Phi(a,b) \neq \emptyset$, and the following properties hold:

1. *the level sets $K_\alpha^\Phi(a,b)$ satisfy the conditional variational principle*

$$\dim_u K_\alpha^\Phi(a,b) = \sup\bigg\{\frac{h_\mu(\Phi)}{\int_\Lambda u d\mu} : \mu \in \mathcal{M}_{erg}(\Phi) \textit{ and } \mathcal{P}(\mu) = \alpha\bigg\};$$

2. $\dim_u K_\alpha^\Phi(a,b) = \inf\{T_u(q) : q \in \mathbb{R}\}$, *where $T_u(q)$ is defined by $T_u(q) = \inf\{t \in \mathbb{R} : P(q(a - \alpha b) - tu) \leq 0\}$;*

3. *for each $\varepsilon > 0$ there exists an ergodic measure μ_α supported on $K_\alpha^\Phi(a,b)$ such that*

$$\bigg|\dim_u \mu_\alpha = \frac{h_{\mu_\alpha}(\Phi)}{\int_\Lambda u d\mu_\alpha} - \dim_u K_\alpha^\Phi(a,b)\bigg| < \varepsilon$$

4. *the spectrum $\alpha \mapsto \dim_u K_\alpha^\Phi(a,b)$ is continuous on $\operatorname{int} \mathcal{P}(\mathcal{M}(\Phi))$.*

Notice that here we no longer require that $\operatorname{span}\{a, b, \overline{u}\} \subset \mathcal{E}(X)$ as in [BH21b]. Moreover, this result extends Theorem 9 in [BH21b] to asymptotically additive families.

The proof will be divided in several steps.

First, let us recall some definitions and introduce some nomenclature used in this section. Let X be a compact metric space, $T : X \to X$ a continuous map. We write

$\mathcal{M}(T)$ to denote the space of T-invariant measures, $\mathcal{M}_{erg}(T)$ the space of T-invariant ergodic measures, and $C(X)$ the space of continuous real valued functions defined on X. Let $\varphi, \psi \in C(X)$, where $\int_X \psi d\mu \geq 0$ for all $\mu \in \mathcal{M}(T)$ with equality only permitted if $\int_X \varphi d\mu \neq 0$. We consider the map $\mathcal{P} : \mathcal{M}(T) \to \mathbb{R}$ given by

$$\mathcal{P}_T(\mu) := \frac{\int_X \varphi d\mu}{\int_X \psi d\mu}.$$

We also define the level sets for φ, ψ by

$$K_\alpha^T(\varphi, \psi) := \left\{ x \in X : \lim_{n\to\infty} \frac{S_n\varphi(x)}{S_n\psi(x)} = \alpha \right\},$$

where $S_n h = \sum_{k=0}^{n-1} h \circ T^k$ for every function $h : X \to \mathbb{R}$.

Theorem 12 ([Cli13, Theorem 3.3])**.** *Let X be a compact metric space and $T : X \to X$ be a continuous map such that the map $\mu \mapsto h_\mu(T)$ is upper semicontinuous and $h(T) < \infty$. Suppose that there is a dense subspace $D \subset C(X)$ such that every $\varphi \in D$ has a unique equilibrium measure. Let $\varphi, \psi \in C(X)$ be such that $\int_X \psi d\mu \geq 0$ for all $\mu \in \mathcal{M}(T)$ with equality only permitted if $\int_X \varphi d\mu \neq 0$. Then $K_\alpha^T(\varphi, \psi) = \emptyset$ for every $\alpha \notin \mathcal{P}_T(\mathcal{M}(T))$, while for every $\alpha \in \operatorname{int} \mathcal{P}_T(\mathcal{M}(T))$ we have that*

1. *the level sets $K_\alpha^T(\varphi, \psi)$ satisfy the conditional variational principle*

$$\dim_u K_\alpha^T(\varphi, \psi) = \sup\left\{ \frac{h_\mu(T)}{\int_X u d\mu} : \mu \in \mathcal{M}(T) \quad \text{and} \quad \mathcal{P}_T(\mu) = \alpha \right\};$$

2. *$\dim_u K_\alpha^T(\varphi, \psi) = \inf\{T_u(q) : q \in \mathbb{R}\}$, where $T_u(q)$ is defined by $T_u(q) = \inf\{t \in \mathbb{R} : P_{classic}(q(\varphi - \alpha\psi) - tu) \leq 0\}$;*

3. *For each $\varepsilon > 0$ there exists an ergodic measure μ supported on $K_\alpha^T(\varphi, \psi))$ such that*

$$\left| \frac{h_\mu(T)}{\int_X u d\mu} - \dim_u K_\alpha^T(\varphi, \psi) \right| < \varepsilon.$$

Moreover, by Proposition 2.14 in [Cli13], one can see that the map

$$\alpha \to \dim_u K_\alpha^T(\varphi, \psi)$$

is continuous on $\operatorname{int} \mathcal{P}_T(\mathcal{M}(T))$.

We recall that a sequence of functions $\mathcal{F} = (f_n)_{n\geq 1}$ is asymptotically additive (with respect to T) if for each $\varepsilon > 0$ there exists a function f such that

$$\limsup_{n\to\infty} \frac{1}{n} \|f_n - S_n f\|_\infty < \varepsilon.$$

We also recall that a sequence $\mathcal{F} = (f_n)_{n\geq 1}$ is almost additive (with respect to T) if there exists $C > 0$ such that

$$-C + f_m(x) + f_n(T^m(x)) \leq f_{m+n}(x) \leq f_m(x) + f_n(T^m(x)) + C$$

for every $x \in X$ and all $m, n \geq 1$.

Let $\mathcal{F} = (f_n)_{n\geq 0}$ and $\mathcal{G} = (g_n)_{n\geq 0}$ be two almost additive sequences of continuous functions where $\lim_{n\to\infty} \frac{1}{n}\int_X g_n d\mu \geq 0$ for all $\mu \in \mathcal{M}(T)$ with equality only permitted if $\lim_{n\to\infty} \frac{1}{n}\int_X f_n d\mu \neq 0$.

Consider level sets

$$K_\alpha^T(\mathcal{F}, \mathcal{G}) := \left\{ x \in X : \lim_{n\to\infty} \frac{f_n(x)}{g_n(x)} = \alpha \right\},$$

and also the map $\mathcal{Q}_T : \mathcal{M}(T) \to \mathbb{R}$ defined by

$$\mathcal{Q}_T(\mu) = \lim_{n\to\infty} \frac{\int_X f_n d\mu}{\int_X g_n d\mu}.$$

Corollary 13. *Let X be a compact metric space and $T : X \to X$ a map satisfying the conditions in Theorem 12. Then $K_\alpha^T(\mathcal{F}, \mathcal{G}) = \emptyset$ for every $\alpha \notin \mathcal{Q}_T(\mathcal{M}(T))$, while for every $\alpha \in \operatorname{int} \mathcal{Q}_T(\mathcal{M}(T))$ we have that*

1. *the level sets $K_\alpha^T(\mathcal{F}, \mathcal{G})$ satisfy the conditional variational principle*

$$\dim_u K_\alpha^T(\mathcal{F}, \mathcal{G}) = \sup\left\{ \frac{h_\mu(T)}{\int_X u d\mu} : \mu \in \mathcal{M}(T) \quad \text{and} \quad \mathcal{Q}_T(\mu) = \alpha \right\};$$

2. *$\dim_u K_\alpha^T(\mathcal{F}, \mathcal{G}) = \inf\{T_u(q) : q \in \mathbb{R}\}$, where $T_u(q)$ is defined by $T_u(q) = \inf\{t \in \mathbb{R} : P(q(\mathcal{F} - \alpha\mathcal{G}) - tu) \leq 0\}$;*

3. *For each $\varepsilon > 0$ there exists an ergodic measure μ supported on $K_\alpha^T(\mathcal{F}, \mathcal{G})$ such that*

$$\left| \frac{h_\mu(T)}{\int_X u d\mu} - \dim_u K_\alpha^T(\mathcal{F}, \mathcal{G}) \right| < \varepsilon.$$

4. *the function $\alpha \to \dim_u K_\alpha^T(\mathcal{F}, \mathcal{G})$ is continuous on $\operatorname{int} \mathcal{Q}_T(\mathcal{M}(T))$.*

Proof. By Theorem 1.2 in [Cun20] there exist $f, g \in C(X)$ such that

$$\lim_{n\to\infty} \frac{1}{n}\|f_n - S_n f\|_\infty = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n}\|g_n - S_n g\|_\infty = 0.$$

By the variational principle for almost additive sequence of continuous functions, one can see that

$$P(q(\mathcal{F} - \alpha\mathcal{G}) - su) = P_{classic}(q(f - \alpha g) - su)$$

for every q, α and $s \in \mathbb{R}$. Moreover, $\mathcal{P}_T(\mu) = \mathcal{Q}_T(\mu)$ for all $\mu \in \mathcal{M}(T)$, and $K_\alpha^T(\mathcal{F}, \mathcal{G}) = K_\alpha^T(f, g)$ for each $\alpha \in \mathbb{R}$. Hence, the result now follows directly from Theorem 12. □

We say that a continuous map $T : X \to X$ on a compact metric space X (or a continuous flow Φ on X) has *entropy density of ergodic measures* if for every invariant measure μ there exist ergodic measures ν_n for $n \in \mathbb{N}$ such that $\nu_n \to \mu$ in the weak* topology and $h_{\nu_n}(T) \to h_\mu(T)$ (or $h_{\nu_n}(\Phi) \to h_\mu(\Phi)$) when $n \to \infty$.

In order to give some examples having entropy density of ergodic measures, we will first recall a few notions. Given $\delta > 0$, we say that T has *weak specification at scale* δ if there exists $\gamma \in \mathbb{N}$ such that for every $(x_1, n_1), \dots, (x_k, n_k) \in X \times \mathbb{N}$ there are $y \in X$ and times $\gamma_1, \dots, \gamma_{k-1} \in \mathbb{N}$ such that $\gamma_i \le \gamma$ and

$$d_{n_i}(T^{s_{i-1}+\gamma_{i-1}}(y), x_i) < \delta \quad \text{for } i = 1, \dots, k,$$

where $s_i = \sum_{i=1}^{i} n_i + \sum_{i=1}^{i-1} \gamma_i$ with $n_0 = \gamma_0 = 0$. When one can take $\gamma_i = \gamma$ for $i = 1, \dots, k-1$, we say that T has *specification at scale* δ. Finally, we say that T has *weak specification* if it has weak specification at every scale δ and, analogously, we say that T has *specification* if it has specification at every scale δ.

It was shown in [EKW94] and [PS05] that mixing subshifts of finite type and mixing locally maximal hyperbolic sets have entropy density of ergodic measures. More recently, it was shown in [CLT20] that a continuous map $T \colon X \to X$ on a compact metric space with the weak specification property such that the entropy map $\mu \mapsto h_\mu(T)$ is upper semicontinuous, has entropy density of ergodic measures. Some examples include expansive maps with specification or with weak specification, topologically transitive locally maximal hyperbolic sets for diffeomorphisms, and transitive topological Markov chains.

We can also introduce the same notions for flows. Let $\Phi = (\phi_t)_{t \in \mathbb{R}}$ be a continuous flow on a compact metric space X. We say that Φ has *weak specification at scale* $\delta > 0$ if there exists $\gamma > 0$ such that for every finite collection of orbit segments $\{(x_i, t_i)\}_{i=1}^k$, there exists a point $y \in X$ and a sequence of *transition times* $\gamma_1, ..., \gamma_{k-1} \in [0, \gamma]$ such that

$$d_{t_j}(\phi_{s_{j-1}+\gamma_{j-1}}(y), x_j) < \delta \quad \text{for} \quad j = 1, ..., k,$$

where $s_j = \sum_{i=1}^{j} t_i + \sum_{i=1}^{j-1} \gamma_i$ and $s_0 = \gamma_0 = 0$. We say that Φ has *weak specification* if it has weak specification at every scale δ. When, for every scale $\delta > 0$, we can take the approximating orbit $y \in X$ to be periodic, and the transition times γ_i close to γ, we say that Φ has *specification*. For a proper definition, see for example [Bow72]. It was also proved in [CLT20] that every expansive flow with the weak specification property has entropy density of ergodic measures. In particular, locally maximal hyperbolic sets for C^1 topologically mixing flows are expansive and have the specification property, that is, they have entropy density of ergodic measures.

Proof of Theorem 11. We are going to use the Markov partitions introduced in subsection 2.2. Let us start proving the result for almost additive families a and b.

Let's consider the following level sets

$$K_\alpha^\Phi(a,b) := \left\{ x \in \Lambda : \lim_{t \to \infty} \frac{a_t(x)}{b_t(x)} = \alpha \right\}, \quad K_\alpha^{T_Z}(c,d) := \left\{ x \in Z : \lim_{n \to \infty} \frac{c_n(x)}{d_n(x)} = \alpha \right\},$$

where $c = (c_n)_{n\in\mathbb{N}}$ and $d = (d_n)_{n\in\mathbb{N}}$ are almost additive sequences of continuous functions given by $c_n(x) = a_{\tau_n(x)}(x)$ and $d_n(x) = b_{\tau_n(x)}(x)$ (see Lemma 3.1 in [BH21a]). By Lemma 3.4 in [BH21a], for every Φ-invariant measure μ induced by an ergodic T_Z-invariant measure ν, we have that

$$\mathcal{P}(\mu) := \lim_{t\to+\infty} \frac{\int_X a_t d\mu}{\int_X b_t d\mu} = \lim_{n\to+\infty} \frac{\int_X c_n d\nu}{\int_X b_n d\nu} := \mathcal{Q}_{T_Z}(\nu). \tag{28}$$

We also recall that a T_Z-invariant measure ν is ergodic if and only if the induced Φ-invariant measure μ is ergodic (see identity (11)).

It follows from Proposition 8, Lemma 14 and Lemma 15 in [BH21b] that

$$\begin{aligned} \dim_u K_\alpha^\Phi(a,b) &= \dim_u\{\phi_s(x) \in \Lambda : x \in K_\alpha(c,d) \text{ and } s \in [0,\tau(x)]\} \\ &= \inf\{\beta \in \mathbb{R} : N_\beta(K_\alpha(c,d)) = 0\} = \dim_{I_u} K_\alpha^{T_Z}(c,d). \end{aligned} \tag{29}$$

Now we are going to check the conditions to apply Corollary 13 for the map T_Z and the sequences c and d. In fact, since Φ is expansive, one can verify that T_Z is also expansive, which implies that $\nu \mapsto h_\nu(T_Z)$ is upper semicontinuous. Moreover, by Abramov's entropy formula (identity (12)), we have

$$h_\nu(T_Z) \le h_\mu(\Phi) \sup \tau \le h(\Phi) \sup \tau < \infty$$

for every $\nu \in \mathcal{M}(T_Z)$, and then $h(T_Z) < \infty$. Letting D be the space of Hölder continuous functions, one can see that D is dense in the space of continuous functions $\varphi : Z \to \mathbb{R}$, and every Hölder continuous function has a unique equilibrium measure with respect to the map T_Z.

By hypothesis, $\lim_{t\to\infty} \frac{1}{t} \int_\Lambda b_t d\mu \ge 0$ for all $\mu \in \mathcal{M}(\Phi)$ with equality only permitted when $\lim_{t\to\infty} \frac{1}{t} \int_\Lambda a_t d\mu \ne 0$. It follows again from Lemma 3.4 in [BH21a] that

$$\frac{\lim_{n\to\infty} \frac{1}{n} \int_Z c_n d\nu}{\int_Z \tau d\nu} = \lim_{t\to\infty} \frac{1}{t} \int_\Lambda a_t d\mu \quad \text{and}$$

$$\frac{\lim_{n\to\infty} \frac{1}{n} \int_Z d_n d\nu}{\int_Z \tau d\nu} = \lim_{t\to\infty} \frac{1}{t} \int_\Lambda b_t d\mu.$$

for every Φ-invariant ergodic measure μ induced by an ergodic T_Z-invariant measure ν. Since $\int_Z \tau d\nu > 0$ for all $\nu \in \mathcal{M}(T_Z)$, we obtain that $\lim_{n\to\infty} \frac{1}{n} \int_Z d_n d\nu \ge 0$ for all $\nu \in \mathcal{M}_{erg}(T_Z)$ and, in particular, that

$$\lim_{n\to\infty} \frac{1}{n} \int_Z d_n d\nu = 0 \quad \text{implies} \quad \lim_{n\to\infty} \frac{1}{n} \int_Z c_n d\nu \ne 0 \quad \text{for all } \nu \in \mathcal{M}_{erg}(T_Z). \tag{30}$$

Now observe that $T_Z : Z \to Z$ is topologically mixing and is conjugated to a topological Markov chain $\sigma : \Sigma_A \to \Sigma_A$ (see subsection 2.2). From this, one can see that T_Z has entropy density of ergodic measures which, in particular, implies that $\mathcal{M}_{erg}(T_Z)$ is dense in $\mathcal{M}(T_Z)$. Let $\eta \in \mathcal{M}(T_Z)$. By the density of $\mathcal{M}_{erg}(T_Z)$ in $\mathcal{M}(T_Z)$, given $\varepsilon > 0$ there exists $\nu \in \mathcal{M}_{erg}(T_Z)$ such that

$$\int_Z \frac{1}{n} d_n d\eta > \int_Z \frac{1}{n} d_n d\nu - \varepsilon \quad \text{for every } n \geq 1.$$

Then

$$\lim_{n\to\infty} \frac{1}{n} \int_Z d_n d\eta \geq \lim_{n\to\infty} \frac{1}{n} \int_Z d_n d\nu - \varepsilon \geq -\varepsilon.$$

Since the measure η and $\varepsilon > 0$ are arbitrary, we conclude that $\lim_{n\to\infty} \frac{1}{n} \int_Z d_n d\nu \geq 0$ for every $\nu \in \mathcal{M}(T_Z)$.

Now fix any measure $\eta \in \mathcal{M}(T)$. Given $\varepsilon > 0$, the entropy density of ergodic measures implies the existence of $\nu \in \mathcal{M}_{erg}(T_Z)$ such that

$$\left| \lim_{n\to\infty} \frac{1}{n} \int_Z d_n d\eta - \lim_{n\to\infty} \frac{1}{n} \int_Z d_n d\nu \right| \leq \varepsilon \quad \text{and} \quad \left| \lim_{n\to\infty} \frac{1}{n} \int_Z c_n d\eta - \lim_{n\to\infty} \frac{1}{n} \int_Z c_n d\nu \right| \leq \varepsilon. \tag{31}$$

Now suppose that

$$\lim_{n\to\infty} \frac{1}{n} \int_Z d_n d\eta = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n} \int_Z c_n d\eta = 0.$$

Then it follows from (31) that

$$\left| \lim_{n\to\infty} \frac{1}{n} \int_Z d_n d\nu \right| \leq \varepsilon \quad \text{and} \quad \left| \lim_{n\to\infty} \frac{1}{n} \int_Z c_n d\nu \right| \leq \varepsilon.$$

By the arbitrariness of ε, this contradicts (30). Hence, since $\eta \in \mathcal{M}(T_Z)$ is also arbitrary, we conclude in particular that

$$\lim_{n\to\infty} \frac{1}{n} \int_Z d_n d\eta = 0 \quad \text{implies} \quad \lim_{n\to\infty} \frac{1}{n} \int_Z c_n d\eta \neq 0 \quad \text{for all } \eta \in \mathcal{M}(T_Z),$$

as desired. Now we finally are in the conditions of applying Corollary 13.

Let $\mu \in \mathcal{M}(T_Z)$ with $\mathcal{Q}_{T_Z}(\mu) = \alpha$. By applying item (1) of Corollary 13 to the map $T_Z : Z \to Z$ and the sequences c and d, we have

$$\dim_{I_u} K_\alpha^{T_Z}(c, d) \geq \frac{h_\mu(T_Z)}{\int_Z I_u d\mu}.$$

Given any $\varepsilon > 0$, by item (3) of Corollary 13, there exists $\nu \in \mathcal{M}_{erg}(T_Z)$ with $\mathcal{Q}_{T_Z}(\nu) = \alpha$ such that

$$\frac{h_\nu(T_Z)}{\int_Z I_u d\nu} > \dim_{I_u} K_\alpha^{T_Z}(c, d) - \varepsilon \geq \frac{h_\mu(T_Z)}{\int_Z I_u d\mu} - \varepsilon.$$

Since the measure μ and $\varepsilon > 0$ are arbitrary, we obtain that

$$\sup\left\{\frac{h_\nu(T_Z)}{\int_X I_u d\nu} : \nu \in \mathcal{M}_{erg}(T_Z) \quad \text{and} \quad \mathcal{Q}_{T_Z}(\nu) = \alpha\right\} \geq \dim_{I_u} K_\alpha^{T_Z}(c,d).$$

Then, from the fact that $\mathcal{M}_{erg}(T_Z) \subset \mathcal{M}(T_Z)$, we conclude that

$$\dim_{I_u} K_\alpha^{T_Z}(c,d) = \sup\left\{\frac{h_\nu(T_Z)}{\int_X I_u d\nu} : \nu \in \mathcal{M}_{erg}(T_Z) \quad \text{and} \quad \mathcal{Q}_{T_Z}(\nu) = \alpha\right\}. \tag{32}$$

Now it follows from (11), (12) and (28) that for each $\nu \in \mathcal{M}_{erg}(T_Z)$ with $\mathcal{Q}_{T_Z}(\nu) = \alpha$, the induced measure $\eta \in \mathcal{M}_{erg}(\Phi)$ is such that $\mathcal{P}(\eta) = \alpha$, and

$$\frac{h_\eta(\Phi)}{\int_\Lambda u d\eta} = \left(\frac{h_\nu(T_Z)}{\int_Z \tau d\nu}\right)\left(\frac{1}{\int_\Lambda u d\eta}\right) = \left(\frac{h_\nu(T_Z)}{\int_Z \tau d\nu}\right)\left(\frac{\int_Z \tau d\nu}{\int_Z I_u d\nu}\right) = \frac{h_\nu(T_Z)}{\int_Z I_u d\nu}. \tag{33}$$

Hence, it follows from (29) and (32) that

$$\dim_u K_\alpha^\Phi(a,b) = \sup\left\{\frac{h_\mu(\Phi)}{\int_\Lambda u d\mu} : \mu \in \mathcal{M}_{erg}(\Phi) \quad \text{and} \quad \mathcal{P}(\mu) = \alpha\right\},$$

and this proves the item (1) of Theorem 11.

Now let us prove the item (2). For each $\mu \in \mathcal{M}_{erg}(\Phi)$ induced by $\nu \in \mathcal{M}_{erg}(T_Z)$, we have

$$\begin{aligned} &h_\mu(\Phi) + \lim_{t\to\infty} \frac{1}{t} \int_\Lambda q(a_t - \alpha b_t) d\mu - s \int_\Lambda u d\mu \\ &= \frac{h_\nu(T_Z)}{\int_Z \tau d\nu} + \frac{\lim_{n\to\infty} \int_Z q(c_n - \alpha d_n) d\nu}{\int_Z \tau d\nu} - \frac{s \int_Z I_u d\nu}{\int_Z \tau d\nu} \end{aligned}$$

for every $\alpha, q, s \in \mathbb{R}$. Since $0 < \inf \tau := \inf_{x\in\Lambda} \tau(x) \leq \inf_{x\in Z} \tau(x)$, we have

$$\begin{aligned} &h_\mu(\Phi) + \lim_{t\to\infty} \frac{1}{t} \int_\Lambda q(a_t - \alpha b_t) d\mu - s \int_\Lambda u d\mu \\ &\leq h_\nu(T_Z) + \lim_{n\to\infty} \int_Z [q(c_n - \alpha d_n) - s I_u] d\nu \left(\frac{1}{\inf \tau}\right) \\ &\leq P(q(c - \alpha d) - s I_u)\left(\frac{1}{\inf \tau}\right), \end{aligned}$$

which implies that

$$P(q(a - \alpha b) - su) \leq P(q(c - \alpha d) - s I_u)\left(\frac{1}{\inf \tau}\right)$$

for every $\alpha, q, s \in \mathbb{R}$. Hence, $T_u(q) \leq T_{I_u}(q)$ for every $q \in \mathbb{R}$.

Since $\sup \tau := \sup_{x \in \Lambda} \tau(x) < \infty$, we can follow in an analogous way to obtain that

$$P(q(a - \alpha b) - su) \geq P(q(c - \alpha d) - sI_u)\left(\frac{1}{\sup \tau}\right)$$

for every $\alpha, q, s \in \mathbb{R}$. Then, $T_u(q) \geq T_{I_u}(q)$ for every $q \in \mathbb{R}$. Since it follows from (29) that $\dim_u K_\alpha^\Phi(a,b) = \dim_{I_u} K_\alpha^{T_Z}(c,d)$, the item (2) of Corollary 13 gives that

$$\dim_u K_\alpha^\Phi(a,b) = \dim_{I_u} K_\alpha^{T_Z}(c,d) = \inf\{T_{I_u}(q) : q \in \mathbb{R}\} = \inf\{T_u(q) : q \in \mathbb{R}\}$$

for each $\alpha \in \operatorname{int} \mathcal{P}(\mathcal{M}(\Phi))$, as desired.

In order to prove item (3), we just observe again that

$$\dim_u K_\alpha^\Phi(a,b) = \dim_{I_u} K_\alpha^{T_Z}(c,d)$$

and that we can use (33) together with item (3) of Corollary 13.

Now let us show how to obtain the item (4) using entropy density of ergodic measures. Since Λ is a locally maximal hyperbolic set for the C^1 topologically mixing flow Φ, we have that $\Phi|_\Lambda$ has entropy density of ergodic measures. In particular, the set $\mathcal{M}_{erg}(\Phi)$ is dense in $\mathcal{M}(\Phi)$. Additionally, recall that T_Z also has entropy density of ergodic measures.

By (28) we already know that $\mathcal{P}(\mathcal{M}_{erg}(\Phi)) = \mathcal{Q}_{T_Z}(\mathcal{M}_{erg}(T_Z))$. Since $\overline{\mathcal{M}_{erg}(\Phi)} = \mathcal{M}(\Phi)$, $\overline{\mathcal{M}_{erg}(T_Z)} = \mathcal{M}(T_Z)$, and the maps $\mu \mapsto \mathcal{P}(\mu)$, $\nu \mapsto \mathcal{Q}_{T_Z}(\nu)$ are continuous, we have that

$$\begin{aligned}\mathcal{Q}_{T_Z}(\mathcal{M}(T_Z)) = \mathcal{Q}_{T_Z}(\overline{\mathcal{M}_{erg}(T_Z)}) &= \overline{\mathcal{Q}_{T_Z}(\mathcal{M}_{erg}(T_Z))} \\ &= \overline{\mathcal{P}(\mathcal{M}_{erg}(\Phi))} = \mathcal{P}(\overline{\mathcal{M}_{erg}(\Phi)}) = \mathcal{P}(\mathcal{M}(\Phi)).\end{aligned} \tag{34}$$

It follows from item (4) of Corollary 13 that $\alpha \mapsto \dim_{I_u} K_\alpha^{T_Z}(c,d)$ is continuous on $\operatorname{int} \mathcal{Q}_{T_Z}(\mathcal{M}(T_Z))$. Since $\dim_u K_\alpha^\Phi(a,b) = \dim_{I_u} K_\alpha^{T_Z}(c,d)$, by (34) we conclude that the map $\alpha \mapsto \dim_u K_\alpha^\Phi(a,b)$ is also continuous on $\operatorname{int} \mathcal{P}(\mathcal{M}(\Phi))$, and the Theorem 11 is proved for almost additive families. Since, in particular, the result holds for the additive case, now we can use directly Corollary 3 to complete the proof for asymptotically additive families. ☐

Remark. Notice that one could start proving Theorem 11 for the additive case (without the hypothesis on density of ergodic measures) and after that, apply Theorem 1 to obtain the full result for asymptotically and, consequently, almost additive families. The choice to start proving the result already in the almost additive case is due to some connections to the aforementioned results in the previous works [BH21a] and [BH21b].

As a final note, observe that since we cannot guarantee the uniqueness of equilibrium measures for asymptotically additive families (even for hyperbolic flows), the extension of Climenhaga's results in [Cli13] to include continuous potentials is crucial for our extension to asymptotically additive families.

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