DIMENSION SPECTRA FOR FLOWS: FUTURE AND PAST

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Abstract. We establish a conditional variational principle for the dimension spectrum obtained from almost additive families for a flow on a conformal locally maximal hyperbolic set, simultaneously into the future and into the past.

1. Introduction

Our main aim is to establish a variational principle for the Hausdorff dimension spectrum obtained from almost additive families for a flow. More precisely, the spectrum is obtained computing the Hausdorff dimension of the level sets obtained from the averages (when they exist) of almost additive families into the past and into the future, on a conformal locally maximal hyperbolic set. We note that the conditional variational principle for the dimension spectrum cannot be obtained from separate results into the future and into the past, at least without further modifications. Instead, we construct noninvariant measures concentrated on each level with the appropriate pointwise dimension that then allow us to obtain the conditional variational principle.

We describe briefly the context of our work. The topological pressure $P(\phi)$ of a continuous function ϕ with respect to a dynamical system $f\colon X \to X$ was introduced by Ruelle in [14] for expansive maps and by Walters in [16] in the general case. Its variational principle says that

$$P(\phi) = \sup_{\mu} \left(h_\mu(f) + \int_X \phi \, d\mu \right),$$

where the supremum is taken over all f-invariant probability measures μ on X and where $h_\mu(f)$ is the Kolmogorov–Sinai entropy of f with respect to μ. We refer the reader to the books [8, 10, 11, 15] for details and further references. The nonadditive thermodynamic formalism was introduced essentially replacing the topological pressure $P(\phi)$ of a single function ϕ by the topological pressure $P(\Phi)$ of a sequence of continuous functions $\Phi = (\phi_n)_{n\in\mathbb{N}}$ (see [1]). Besides playing a unifying role, the nonadditive thermodynamic formalism has nontrivial applications to the dimension theory and multifractal analysis of dynamical systems. With the same spirit in mind, in [5] we considered a version of the nonadditive topological pressure for almost additive families with respect to a flow.

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A family $a^+ = (a_t^+)_{t\ge 0}$ is said to be *almost additive* (into the future) with respect to a flow $\Phi = (\phi_t)_{t\in\mathbb{R}}$ if there exists a constant $C > 0$ such that

$$-C \le a_{t+s}^+ - a_t^+ - a_s^+ \circ \phi_t \le C \tag{1}$$

for every $t, s > 0$. We showed in [5] that if a^+ is an almost additive family of continuous functions with tempered variation (see Section 2.1) such that

$$\sup_{t\in[0,s]} \|a_t^+\|_\infty < \infty \quad \text{for some } s > 0,$$

then

$$P_\Phi(a^+) = \sup_{\mu\in\mathcal{M}_\Phi} \left(h_\mu(\Phi) + \lim_{t\to\infty} \frac{1}{t} \int_X a_t^+ \, d\mu \right), \tag{2}$$

where $\mathcal{M}_\Phi$ is the set of all Φ-invariant probability measures on X. We say that a Φ-invariant measure μ on X is an *equilibrium measure* for a^+ (with respect to Φ) if the supremum in (2) is attained at μ, that is, if

$$P_\Phi(a^+) = h_\mu(\Phi) + \lim_{t\to\infty} \frac{1}{t} \int_X a_t^+ \, d\mu.$$

We also showed that if Λ is a hyperbolic set for a topologically mixing C^1 flow Φ and the family a^+ has bounded variation (see Section 4 for the definition), then there exists a unique equilibrium measure for a^+.

In this work we establish a conditional variational principle for the Hausdorff dimension spectrum obtained from almost additive families for a flow on a conformal locally maximal hyperbolic set. Moreover, we consider simultaneously the behaviors into the future and into the past. For simplicity of the exposition, here we formulate only a particular case.

Let $a^+ = (a_t^+)_{t\ge 0}$ be a family of continuous functions on a hyperbolic set Λ that is almost additive into the future (see (1)). Let also $a^- = (a_t^-)_{t\ge 0}$ be an almost additive sequence of continuous functions on Λ that is almost additive into the past, that is, there exists a constant $C > 0$ such that

$$-C \le a_{t+s}^- - a_t^- - a_s^- \circ \phi_{-t} \le C$$

for every $t, s \ge 0$. Given $\alpha, \beta \in \mathbb{R}$, we consider the sets

$$K_\alpha^+ = \left\{ x \in \Lambda : \lim_{t\to\infty} \frac{a_t^+(x)}{t} = \alpha \right\}$$

and

$$K_\beta^- = \left\{ x \in \Lambda : \lim_{t\to\infty} \frac{a_t^-(x)}{t} = \beta \right\}.$$

Our main result is a variational principle for the *dimension spectrum*

$$\mathcal{D}(\alpha, \beta) = \dim_H(K_\alpha^+ \cap K_\beta^-).$$

We also consider the maps $\mathcal{P}^\pm \colon \mathcal{M}_\Phi \to \mathbb{R}$ defined by

$$\mathcal{P}^+(\mu) = \lim_{t\to\infty} \frac{1}{t} \int_\Lambda a_t^+ \, d\mu, \quad \mathcal{P}^-(\mu) = \lim_{t\to\infty} \frac{1}{t} \int_\Lambda a_t^- \, d\mu,$$

as well as the functions

$$\xi_s(x) = \lim_{t\to 0} \frac{\log \|d_x\phi_t | E^s(x)\|}{t}, \quad \xi_u(x) = \lim_{t\to 0} \frac{\log \|d_x\phi_t | E^u(x)\|}{t},$$

where $E^s(x)$ and $E^u(x)$ are the stable and unstable spaces at x.

Theorem 1. *Let Φ be a $C^{1+\varepsilon}$ flow with a locally maximal hyperbolic set Λ such that Φ is conformal and topologically mixing on Λ. Then:*

(1) *if $\alpha \in \operatorname{int} \mathcal{P}^+(\mathcal{M}_\Phi)$ and $\beta \in \operatorname{int} \mathcal{P}^-(\mathcal{M}_\Phi)$, then*

$$\mathcal{D}(\alpha, \beta) = \max\left\{ \frac{h_\mu(\Phi)}{\int_\Lambda \xi_u \, d\mu} : \mu \in \mathcal{M}_\Phi \text{ and } \mathcal{P}^+(\mu) = \alpha \right\} + \max\left\{ \frac{h_\mu(\Phi)}{-\int_\Lambda \xi_s \, d\mu} : \mu \in \mathcal{M}_\Phi \text{ and } \mathcal{P}^-(\mu) = \beta \right\} + 1;$$

(2) *$\mathcal{D}$ is continuous on $\operatorname{int} \mathcal{P}^+(\mathcal{M}_\Phi) \times \operatorname{int} \mathcal{P}^-(\mathcal{M}_\Phi)$.*

A corresponding result for discrete time was obtained earlier in [4] for almost additive sequences, on a conformal locally maximal hyperbolic set for a diffeomorphism. To the possible extent we follow their approach by using Markov systems for the hyperbolic set and the associated symbolic dynamics along the stable and unstable invariant manifolds.

2. Thermodynamic formalism

2.1. **Topological pressure.** Let $\Phi = (\phi_t)_{t \in \mathbb{R}}$ be a continuous flow on a compact metric space (X, d). Moreover, let $a = (a_t)_{t \geq 0}$ be a family of continuous functions $a_t \colon X \to \mathbb{R}$ with *tempered variation*, that is, such that

$$\lim_{\varepsilon \to 0} \overline{\lim_{t \to \infty}} \frac{\gamma_t(a, \varepsilon)}{t} = 0, \tag{3}$$

where

$$\gamma_t(a, \varepsilon) = \sup\bigl\{ |a_t(y) - a_t(x)| : y \in B_t(x, \varepsilon) \text{ for some } x \in X \bigr\}$$

and

$$B_t(x, \varepsilon) = \bigl\{ y \in X : d(\phi_s(y), \phi_s(x)) < \varepsilon \text{ for } s \in [0, t] \bigr\}.$$

Given $\varepsilon > 0$, we say that $\Gamma \subset X \times \mathbb{R}_0^+$ *covers* a set $Z \subset X$ if

$$\bigcup_{(x,t) \in \Gamma} B_t(x, \varepsilon) \supset Z$$

and we write

$$a(x, t, \varepsilon) = \sup\bigl\{ a_t(y) : y \in B_t(x, \varepsilon) \bigr\}$$

for $(x, t) \in \Gamma$. For each $Z \subset X$ and $\alpha \in \mathbb{R}$, let

$$M(Z, a, \alpha, \varepsilon) = \lim_{T \to \infty} \inf_\Gamma \sum_{(x,t) \in \Gamma} \exp(a(x, t, \varepsilon) - \alpha t),$$

with the infimum taken over all countable sets $\Gamma \subset X \times [T, +\infty)$ covering Z. When α goes from $-\infty$ to $+\infty$, the map $\alpha \mapsto M(Z, a, \alpha, \varepsilon)$ jumps from $+\infty$ to 0 at a unique value and so one can define

$$P_\Phi(a|_Z, \varepsilon) = \inf\bigl\{ \alpha \in \mathbb{R} : M(Z, a, \alpha, \varepsilon) = 0 \bigr\}.$$

Moreover, the limit

$$P_\Phi(a|_Z) = \lim_{\varepsilon \to 0} P_\Phi(a|_Z, \varepsilon)$$

exists and is called the *topological pressure* of the family a on the set Z. For simplicity of the notation, we shall also write $P_\Phi(a|_X) = P_\Phi(a)$.

The classical notion of topological pressure for a flow corresponds to consider a family of continuous functions $a = (a_t)_{t\geq 0}$ defined by

$$a_t(x) = \int_0^t b(\phi_s(x))\,ds$$

for some continuous function $b\colon X \to \mathbb{R}$. One can easily verify that (3) holds for this family and we write $P(b) = P_\Phi(a)$.

2.2. **u-dimension for flows.** Given a continuous function $u\colon X \to \mathbb{R}^+$, we consider the family of continuous functions $\bar{u} = (u_t)_{t\geq 0}$ defined by

$$u_t(x) = \int_0^t u(\phi_s(x))\,ds$$

for every $x \in X$ and $t \geq 0$. For each $Z \subset X$ and $\alpha \in \mathbb{R}$, let

$$N(Z, u, \alpha, \varepsilon) = \lim_{T\to\infty} \inf_{\Gamma} \sum_{(x,t)\in\Gamma} e^{-\alpha u(x,t,\varepsilon)},$$

with the infimum taken over all countable sets $\Gamma \subset X \times [T, +\infty)$ covering Z. Finally, let

$$\dim_{u,\varepsilon} Z = \inf\bigl\{\alpha \in \mathbb{R} : N(Z, u, \alpha, \varepsilon) = 0\bigr\}.$$

The limit

$$\dim_u Z := \lim_{\varepsilon\to 0} \dim_{u,\varepsilon} Z$$

exists and is called the *u-dimension* of the set Z (with respect to the flow Φ). One can easily verify that $\dim_u Z = \alpha$, where α is the unique root of the equation $P_\Phi(-\alpha\bar{u}|_Z) = 0$.

3. Flows and hyperbolicity

3.1. **Hyperbolic sets.** Let $\Phi = (\phi_t)_{t\in\mathbb{R}}$ be a C^1 flow on a smooth manifold M with distance d. A compact Φ-invariant set $\Lambda \subset M$ is said to be a *hyperbolic set for* Φ if there exist a splitting

$$T_\Lambda M = E^s \oplus E^u \oplus E^\Phi$$

and constants $c > 0$ and $\lambda \in (0, 1)$ such that for each $x \in \Lambda$:

(1) the vector $(d/dt)\phi_t(x)|_{t=0}$ generates $E^\Phi(x)$;
(2) for each $t \in \mathbb{R}$ we have
$$d_x\phi_t E^s(x) = E^s(\phi_t(x)) \quad \text{and} \quad d_x\phi_t E^u(x) = E^u(\phi_t(x));$$
(3)
$$\|d_x\phi_t v\| \leq c\lambda^t \|v\| \quad \text{for } v \in E^s(x),\ t > 0;$$
(4)
$$\|d_x\phi_{-t} v\| \leq c\lambda^t \|v\| \quad \text{for } v \in E^u(x),\ t > 0.$$

Given a hyperbolic set Λ and $\varepsilon > 0$, for each $x \in \Lambda$ let $V^s(x)$ and $V^u(x)$ be, respectively, the connected components of the sets

$$A^s(x) = \bigl\{y \in B(x, \varepsilon) : d(\phi_t(y), \phi_t(x)) \to 0 \text{ when } t \to +\infty\bigr\}$$

and

$$A^u(x) = \bigl\{y \in B(x, \varepsilon) : d(\phi_t(y), \phi_t(x)) \to 0 \text{ when } t \to -\infty\bigr\}$$

containing x. The sets $V^s(x)$ and $V^u(x)$ are called, respectively, *stable* and *unstable local manifolds* at x (of size ε). We have the following properties:

(1) $T_xV^s(x) = E^s(x)$ and $T_xV^u(x) = E^u(x)$;

(2) for each $t > 0$ we have

$$\phi_t(V^s(x)) \subset V^s(\phi_t(x)) \quad \text{and} \quad \phi_{-t}(V^u(x)) \subset V^u(\phi_{-t}(x));$$

(3) there exist $d > 0$ and $\mu \in (0, 1)$ such that

$$d(\phi_t(y), \phi_t(x)) \le d\mu^t d(y, x) \quad \text{for } t > 0,\ y \in V^s(x) \tag{4}$$

and

$$d(\phi_{-t}(y), \phi_{-t}(x)) \le d\mu^t d(y, x) \quad \text{for } t > 0,\ y \in V^u(x).$$

Given a locally maximal hyperbolic set Λ (that is, a hyperbolic set Λ such that $\Lambda = \bigcap_{t\in\mathbb{R}} \phi_t(U)$ for some open neighborhood U of Λ) and a sufficiently small $\tau > 0$, there exists $\delta > 0$ such that if $x, y \in \Lambda$ are at a distance $d(x, y) \le \delta$, then there exists a unique $t = t(x, y) \in [-\tau, \tau]$ such that

$$[x, y] := V^s(\phi_t(x)) \cap V^u(y)$$

is a single point of Λ.

3.2. **Conformal flows.** We say that a C^1 flow Φ is *conformal* on a hyperbolic set Λ if there exist continuous functions $P^s, P^u \colon \Lambda \times \mathbb{R} \to \mathbb{R}$ such that

$$d_x\phi_t|E^s(x) = P^s(x, t)I^s(x, t) \quad \text{and} \quad d_x\phi_t|E^u(x) = P^u(x, t)I^u(x, t)$$

for every $x \in \Lambda$ and $t \in \mathbb{R}$, where

$$I^s(x, t) \colon E^s(x) \to E^s(\phi_t(x)) \quad \text{and} \quad I^u(x, t) \colon E^u(x) \to E^u(\phi_t(x))$$

are isometries. For example, if

$$\dim E^s(x) = \dim E^u(x) = 1 \quad \text{for } x \in \Lambda,$$

then the flow is conformal on Λ. Following [12] we define:

$$\begin{aligned} \xi_s(x) &:= \frac{\partial}{\partial t} \log |P^s(x, t)|_{t=0} \\ &= \frac{\partial}{\partial t} \log \|d_x\phi_t|E^s(x)\|_{t=0} \\ &= \lim_{t\to 0} \frac{\log \|d_x\phi_t|E^s(x)\|}{t} \end{aligned} \tag{5}$$

and

$$\begin{aligned} \xi_u(x) &:= \frac{\partial}{\partial t} \log |P^u(x, t)|_{t=0} \\ &= \frac{\partial}{\partial t} \log \|d_x\phi_t|E^u(x)\|_{t=0} \\ &= \lim_{t\to 0} \frac{\log \|d_x\phi_t|E^u(x)\|}{t}. \end{aligned} \tag{6}$$

Since the flow Φ is of class C^1, using 2-norms one can write

$$\begin{aligned}\lim_{t\to 0}\frac{\log\|d_x\phi_t|E^s(x)\|}{t} &= \lim_{t\to 0}\frac{\log(\|d_x\phi_t|E^s(x)\|^2)}{2t}\\ &= \lim_{t\to 0}\frac{\langle d_x\phi_t|E^s(x), \frac{\partial}{\partial t}(d_x\phi_t|E^s(x))\rangle}{\|d_x\phi_t|E^u(x)\|^2}\\ &= \left\langle \mathrm{Id}|E^s(x), \frac{\partial}{\partial t}(d_x\phi_t|E^s(x))|_{t=0}\right\rangle\end{aligned}$$

and, similarly,

$$\lim_{t\to 0}\frac{\log\|d_x\phi_t|E^u(x)\|}{t} = \left\langle \mathrm{Id}|E^u(x), \frac{\partial}{\partial t}(d_x\phi_t|E^u(x))|_{t=0}\right\rangle.$$

In particular, the functions ξ_s and ξ_u are well defined. Furthermore:

(1) Since the maps $x\mapsto E^s(x)$ and $x\mapsto E^u(x)$ are Hölder continuous, the functions ξ_s and ξ_u are also Hölder continuous.

(2) For an adapted norm $\|\cdot\|$ (that is, a norm for which one can take $c=1$ in the definition of a hyperbolic set), we obtain

$$\xi_s(x) = \lim_{t\to 0^+}\frac{\log\|d_x\phi_t|E^s(x)\|}{t} \le \log\lambda < 0$$

and

$$\xi_u(x) = \lim_{t\to 0^+}\frac{\log\|d_x\phi_t|E^u(x)\|}{t} \ge -\log\lambda > 0$$

for all $x\in\Lambda$.

(3) For every $x\in\Lambda$ and $t\in\mathbb{R}$, it follows from (5) and (6) that

$$\|d_x\phi_t v\| = \|v\|\exp\biggl(\int_0^t \xi_s(\phi_\tau(x))\,d\tau\biggr)\quad \text{for } v\in E^s(x)$$

and

$$\|d_x\phi_t v\| = \|v\|\exp\biggl(\int_0^t \xi_u(\phi_\tau(x))\,d\tau\biggr)\quad \text{for } v\in E^u(x). \tag{7}$$

4. Almost additive families

In this section we introduce the general context of our work: the study of level sets associated with almost additive families of continuous functions. Let $\Phi=(\phi_t)_{t\in\mathbb{R}}$ be a continuous flow on a compact metric space (X,d).

A family $a=(a_t)_{t\ge 0}$ of continuous functions $a_t\colon\Lambda\to\mathbb{R}$ is said to be *almost additive into the future* if there exists a constant $C_1>0$ such that

$$-C_1 \le a_{t+s}(x) - a_t(x) - a_s(\phi_t(x)) \le C_1$$

for every $x\in\Lambda$ and $t,s\ge 0$. Analogously, a family $a=(a_t)_{t\ge 0}$ is said to be *almost additive into the past* if there exists a constant $C_2>0$ such that

$$-C_2 \le a_{t+s}(x) - a_t(x) - a_s(\phi_{-t}(x)) \le C_2$$

for every $x\in\Lambda$ and $t,s\ge 0$. We recall that a family $a=(a_t)_{t\ge 0}$ is said to have *bounded variation* (with respect to the flow Φ) if for every $\kappa>0$ there exists $\varepsilon>0$ such that

$$|a_t(x)-a_t(y)|<\kappa \quad \text{whenever } y\in B_t(x,\varepsilon).$$

We denote by $\mathcal{A}^+$ the set of all families $a = (a_t)_{t\geq 0}$ of continuous functions $a_t\colon \Lambda \to \mathbb{R}$ with bounded variation with respect to the flow Φ that are almost additive into the future and satisfy

$$\sup_{t\in[0,s]} \|a_t\|_\infty < +\infty \quad \text{for some } s > 0. \tag{8}$$

Similarly, we denote by $\mathcal{A}^-$ the set of all families $a = (a_t)_{t\geq 0}$ of continuous functions $a_t\colon \Lambda \to \mathbb{R}$ with bounded variation with respect to the flow $(\phi_{-t})_{t\in\mathbb{R}}$ that are almost additive into the past and satisfy

$$\sup_{t\in[-s,0]} \|a_t\|_\infty < +\infty \quad \text{for some } s > 0.$$

Now consider pairs $(a^+, b^+) \in \mathcal{A}^+ \times \mathcal{A}^+$ and $(a^-, b^-) \in \mathcal{A}^- \times \mathcal{A}^-$ such that

$$\liminf_{t\to\infty} \frac{b_t^\pm(x)}{t} > 0 \quad \text{and} \quad b_t^\pm(x) > 0 \tag{9}$$

for every $x \in \Lambda$ and $t \geq 0$. Given $\alpha, \beta \in \mathbb{R}$, we consider the *level sets*

$$K_\alpha^+ = \left\{ x \in \Lambda : \lim_{t\to\infty} \frac{a_t^+(x)}{b_t^+(x)} = \alpha \right\} \tag{10}$$

and

$$K_\beta^- = \left\{ x \in \Lambda : \lim_{t\to\infty} \frac{a_t^-(x)}{b_t^-(x)} = \beta \right\}. \tag{11}$$

We also consider the sets $K_\alpha^+ \cap K_\beta^-$ that consider simultaneously the asymptotic behaviors into the future and into the past.

It was shown in [6] that if a is an almost additive family of continuous functions (into the future) with tempered variation such that $\sup_{t\in[0,s]} \|a_t\|_\infty < \infty$ for some $s > 0$, then we have the variational principle

$$P_\Phi(a) = \sup_{\mu\in\mathcal{M}_\Phi} \left(h_\mu(\Phi) + \lim_{t\to\infty} \frac{1}{t} \int_\Lambda a_t \, d\mu \right), \tag{12}$$

where $\mathcal{M}_\Phi$ is the set of all Φ-invariant probability measures on Λ and where $h_\mu(\Phi)$ is the Kolmogorov–Sinai entropy of μ. We say that a measure $\mu \in \mathcal{M}_\Phi$ is an *equilibrium measure* for the almost additive family a (with respect to the flow Φ) if the supremum in (12) is attained at μ, that is, if

$$P_\Phi(a) = h_\mu(\Phi) + \lim_{t\to\infty} \frac{1}{t} \int_\Lambda a_t \, d\mu.$$

5. Dimensions along the stable and unstable directions

In this section we obtain formulas for the Hausdorff dimensions of the level sets K_α^+ and K_β^- in (10) and (11) in terms of the topological pressure.

Before proceeding we recall a result of Pesin and Sadovskaya in [12] on the Hausdorff dimensions of a hyperbolic set along the stable and unstable local manifolds. We denote by $\dim_H S$, $\overline{\dim}_B S$ and $\underline{\dim}_B S$, respectively, the Hausdorff dimension, the lower box dimension and the upper box dimension of a set S.

Proposition 2 ([12, Theorem 4.1])**.** *Let Φ be a $C^{1+\varepsilon}$ flow with a locally maximal hyperbolic set Λ such that Φ is conformal and topologically mixing on Λ. For every $x \in \Lambda$ we have*

$$\dim_H(\Lambda \cap V^s(x)) = \overline{\dim}_B(\Lambda \cap V^s(x)) = \underline{\dim}_B(\Lambda \cap V^s(x)) = t_s$$

and

$$\dim_H(\Lambda \cap V^u(x)) = \overline{\dim}_B(\Lambda \cap V^u(x)) = \underline{\dim}_B(\Lambda \cap V^u(x)) = t_u,$$

where t_s and t_u are the unique real numbers such that

$$P(t_s\xi_s) = 0 \quad \textit{and} \quad P(-t_u\xi_u) = 0.$$

The following result describes the Hausdorff dimensions of the level sets K_α^+ and K_β^- in terms of the topological pressure.

Theorem 3. *Let Φ be a $C^{1+\varepsilon}$ flow with a locally maximal hyperbolic set Λ such that Φ is conformal and topologically mixing on Λ and take pairs*

$$(a^+, b^+) \in \mathcal{A}^+ \times \mathcal{A}^+ \quad \textit{and} \quad (a^-, b^-) \in \mathcal{A}^- \times \mathcal{A}^-$$

satisfying (9)*. For each $(\alpha, \beta) \in \mathbb{R}^2$, $x^+ \in K_\alpha^+$ and $x^- \in K_\beta^-$, we have:*

(1)
$$\Lambda \cap V^s(x^+) \subset K_\alpha^+ \quad \textit{and} \quad \Lambda \cap V^u(x^-) \subset K_\beta^-; \tag{13}$$

(2)
$$\begin{aligned} \dim_H K_\alpha^+ &= \dim_H(K_\alpha^+ \cap V^u(x^+)) + t_s + 1 \\ &= \dim_{\xi_u} K_\alpha^+ + t_s + 1 \end{aligned} \tag{14}$$

and

$$\begin{aligned} \dim_H K_\beta^- &= \dim_H(K_\beta^- \cap V^s(x^-)) + t_u + 1 \\ &= \dim_{-\xi_s} K_\beta^- + t_u + 1. \end{aligned} \tag{15}$$

Proof. Since the families a^+ and b^+ have bounded variation, given $\kappa > 0$, there exists $\varepsilon > 0$ such that

$$|a_t^+(y) - a_t^+(z)| < \kappa \quad \text{and} \quad |b_t^+(y) - b_t^+(z)| < \kappa$$

for $y, z \in B_t(x^+, \varepsilon)$. Now take $y, z \in V^s(x^+)$. Provided that y and z are sufficiently close, it follows from (4) that $y, z \in B_t(x^+, \varepsilon)$. We have

$$\begin{aligned} \left| \frac{a_t^+(y)}{b_t^+(y)} - \frac{a_t^+(z)}{b_t^+(z)} \right| &= \left| \frac{a_t^+(y)}{b_t^+(y)} - \frac{a_t^+(z)}{b_t^+(y)} + \frac{a_t^+(z)}{b_t^+(y)} - \frac{a_t^+(z)}{b_t^+(z)} \right| \\ &\le \frac{|a_t^+(y) - a_t^+(z)|}{b_t^+(y)} + |a_t^+(z)| \cdot \left| \frac{1}{b_t^+(y)} - \frac{1}{b_t^+(z)} \right| \\ &= \frac{|a_t^+(y) - a_t^+(z)|}{b_t^+(y)} + |a_t^+(z)| \cdot \frac{|b_t^+(y) - b_t^+(z)|}{b_t^+(y) b_t^+(z)}. \end{aligned}$$

Since a^+ is almost additive, by (8) there exists $K > 0$ such that

$$\|a_t^+(y)\| \le K(1 + t) \quad \text{for all } y \in \Lambda,\ t \ge 0.$$

Moreover, by (9), there exists $C > 0$ such that

$$\|b_t^+(y)\| \ge Ct \quad \text{for all } y \in \Lambda,\ t \ge 0.$$

Therefore,

$$\left|\frac{a_t^+(y)}{b_t^+(y)} - \frac{a_t^+(z)}{b_t^+(z)}\right| \le \frac{\kappa}{Ct} + K(1+t)\frac{\kappa}{C^2t^2}$$

and so

$$\lim_{t\to\infty}\left|\frac{a_t^+(y)}{b_t^+(y)} - \frac{a_t^+(z)}{b_t^+(z)}\right| = 0. \tag{16}$$

Now we cover a compact neighborhood of x^+ in $V^s(x^+)$ with sufficiently small balls $B_y := B(y,r)\cap V^s(x^+)$ such that property (16) holds for $z \in B_y$. Taking a finite subcover, it follows readily from (16) that

$$\lim_{t\to\infty}\frac{a_t^+(y)}{b_t^+(y)} = \lim_{t\to\infty}\frac{a_t^+(x^+)}{b_t^+(x^+)} = \alpha$$

for all y in the compact neighborhood of x^+ in $V^s(x^+)$. In other words,

$$\Lambda \cap V^s(x^+) \subset K_\alpha^+ \quad \text{for every } x^+ \in K_\alpha^+.$$

One can establish the second inclusion in (13) in a similar manner.

Since the set K_α^+ is Φ-invariant (see [6]), we have

$$\Lambda \cap \bigcup_{t\in\mathbb{R}} \phi_t(V^s(x^+)) \subset K_\alpha^+.$$

On the other hand, since $\Phi|\Lambda$ is conformal, it follows from results in [9] that the maps

$$x \mapsto E^s(x) \oplus E^\Phi(x) \quad \text{and} \quad x \mapsto E^u(x) \oplus E^\Phi(x)$$

are Lipschitz. This implies that on a sufficiently small open neighborhood of a point $x^+ \in K_\alpha^+$, there exists a Lipschitz map with Lipschitz inverse from K_α^+ onto the product

$$(\Lambda \cap V_I^u(x^+)) \times (K_\alpha^+ \cap V^u(x^+)),$$

where

$$V_I^u(x^+) = \bigcup_{t\in I} \phi_t(V^s(x^+))$$

for some open interval $I \subset \mathbb{R}$ containing zero. Therefore,

$$\dim_H K_\alpha^+ = \dim_H \big[(\Lambda \cap V_I^u(x^+)) \times (K_\alpha^+ \cap V^u(x^+))\big].$$

By Proposition 2, we have

$$\dim_H(\Lambda \cap V_I^u(x^+)) = \overline{\dim}_B(\Lambda \cap V_I^u(x^+)) = t_s + 1. \tag{17}$$

Since

$$\dim_H S_1 + \dim_H S_2 \le \dim_H(S_1 \times S_2) \le \overline{\dim}_B S_1 + \dim_H S_2$$

for any sets $S_1, S_2 \subset \mathbb{R}^n$, it follows from (17) that

$$\dim_H K_\alpha^+ = \dim_H(K_\alpha^+ \cap V^u(x^+)) + t_s + 1,$$

which is the first equality in (14). The first equality in (15) can be obtained in a similar manner.

Now we establish the second equality in (14). Let

$$\xi_u(x,t,\varepsilon) = \sup\big\{a_t(y) : y \in B_t(x,\varepsilon)\big\},$$

where

$$a_t(x) = \int_0^t \xi_u(\phi_s(x))\, ds.$$

Since the function ξ_u is Hölder continuous, it follows from (7) that given $\varepsilon > 0$, there exist constants $k_1, k_2 > 0$ such that

$$k_1 e^{-\gamma \xi_u(x,t,\varepsilon)} \le \left[\operatorname{diam}\left(B_t(x,\varepsilon) \cap V^u(x^+)\right)\right]^\gamma \le k_2 e^{-\gamma \xi_u(x,t,\varepsilon)}$$

for every $x \in \Lambda$, $t > 0$ and $\gamma > 0$. This readily implies that

$$\dim_{\xi_u} S = \dim_H(S \cap V^u(x^+))$$

for any set $S \subset \Lambda$. In particular, taking $S = K_\alpha^+$ we obtain

$$\dim_H(K_\alpha^+ \cap V^u(x^+)) = \dim_{\xi_u} K_\alpha^+.$$

One can obtain in a similar manner a corresponding result for K_β^-. □

6. Conditional variational principle

6.1. **Formulation of the result.** In this section we obtain a conditional variational principle for the *dimension spectrum*

$$\mathcal{D}(\alpha, \beta) = \dim_H(K_\alpha^+ \cap K_\beta^-)$$

obtained from families of continuous functions $(a^\pm, b^\pm) \in \mathcal{A}^\pm \times \mathcal{A}^\pm$. We also consider the functions $\mathcal{P}_\Phi^\pm \colon \mathcal{M}_\Phi \to \mathbb{R}$ defined by

$$\mathcal{P}_\Phi^+(\mu) = \lim_{t\to\infty} \frac{\int_\Lambda a_t^+ \, d\mu}{\int_\Lambda b_t^+ \, d\mu} \quad \text{and} \quad \mathcal{P}_\Phi^-(\mu) = \lim_{t\to\infty} \frac{\int_\Lambda a_t^- \, d\mu}{\int_\Lambda b_t^- \, d\mu},$$

where $\mathcal{M}_\Phi$ is the set of all Φ-invariant probability measures on Λ. The following theorem is the main result of this section.

Theorem 4. *Let Φ be a $C^{1+\varepsilon}$ flow with a locally maximal hyperbolic set Λ such that Φ is conformal and topologically mixing on Λ and take pairs*

$$(a^+, b^+) \in \mathcal{A}^+ \times \mathcal{A}^+ \quad \text{and} \quad (a^-, b^-) \in \mathcal{A}^- \times \mathcal{A}^-$$

satisfying (9)*. Then the following properties hold:*

(1) *if $\alpha \in \operatorname{int} \mathcal{P}_\Phi^+(\mathcal{M}_\Phi)$ and $\beta \in \operatorname{int} \mathcal{P}_\Phi^-(\mathcal{M}_\Phi)$, then*

$$\begin{aligned}\mathcal{D}(\alpha,\beta) &= \dim_H K_\alpha^+ + \dim_H K_\beta^- - \dim_H \Lambda \\ &= \max\left\{\frac{h_\mu(\Phi)}{\int_\Lambda \xi_u \, d\mu} : \mu \in \mathcal{M}_\Phi \text{ and } \mathcal{P}_\Phi^+(\mu) = \alpha\right\} \\ &\quad + \max\left\{\frac{h_\mu(\Phi)}{-\int_\Lambda \xi_s \, d\mu} : \mu \in \mathcal{M}_\Phi \text{ and } \mathcal{P}_\Phi^-(\mu) = \beta\right\} + 1;\end{aligned}$$

(2) *$\mathcal{D}$ is continuous on $\operatorname{int} \mathcal{P}_\Phi^+(\mathcal{M}_\Phi) \times \operatorname{int} \mathcal{P}_\Phi^-(\mathcal{M}_\Phi)$.*

6.2. **Markov systems.** For the proof of Theorem 4 we need the notion of a Markov system and its associated symbolic dynamics. Let $D \subset M$ be an open smooth disk of dimension $\dim M - 1$ transverse to the flow Φ and take $x \in D$. Let $U(x)$ be an open neighborhood of x diffeomorphic to $D \times (-\varepsilon, \varepsilon)$. A closed set $R \subset \Lambda \cap D$ is called a *rectangle* if

$$R = \overline{\operatorname{int} R} \quad \text{and} \quad \pi_D([x, y]) \in R \text{ for } x, y \in R.$$

Now consider rectangles $R_1, \ldots, R_k \subset \Lambda$ such that

$$R_i \cap R_j = \partial R_i \cap \partial R_j \quad \text{for } i \neq j$$

and let $Z = \bigcup_{i=1}^k R_i$. We assume that $\Lambda = \bigcup_{t \in [0,\varepsilon]} \phi_t(Z)$ and that either

$$\phi_t(R_i) \cap R_j = \emptyset \quad \text{for all } t \in [0, \varepsilon]$$

or

$$\phi_t(R_j) \cap R_i = \emptyset \quad \text{for all } t \in [0, \varepsilon]$$

when $i \neq j$. We define the corresponding *transfer function* $\tau \colon \Lambda \to \mathbb{R}_0^+$ by

$$\tau(x) = \min\bigl\{t > 0 : \phi_t(x) \in Z\bigr\}$$

and the *transfer map* $T \colon \Lambda \to Z$ by $T(x) = \phi_{\tau(x)}(x)$. The restriction T_Z of T to Z is invertible and we have $T^n(x) = \phi_{\tau_n(x)}(x)$, where

$$\tau_n(x) = \sum_{i=0}^{n-1} \tau(T^i(x)).$$

The collection $R_1, \ldots, R_k$ is called a *Markov system* for Φ on Λ if

$$T(\operatorname{int}(V^s(x) \cap R_i)) \subset \operatorname{int}(V^s(T(x)) \cap R_j)$$

and

$$T^{-1}(\operatorname{int}(V^u(T(x)) \cap R_j)) \subset \operatorname{int}(V^u(x) \cap R_i)$$

for every $x \in \operatorname{int} T(R_i) \cap \operatorname{int} R_j$ and $i, j = 1, \ldots, k$. By work of Bowen [7] and Ratner [13], any locally maximal hyperbolic set Λ has Markov systems of arbitrarily small diameter.

Given a Markov system $R_1, \ldots, R_k$ for a flow Φ on a locally maximal hyperbolic set Λ, we consider the $k \times k$ matrix A with entries

$$a_{ij} = \begin{cases} 1 & \text{if } \operatorname{int} T(R_i) \cap R_j \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

We also consider the set

$$\Sigma_A = \bigl\{(\cdots i_{-1} i_0 i_1 \cdots) \in \{1, \ldots, k\}^{\mathbb{Z}} : a_{i_n i_{n+1}} = 1 \text{ for } n \in \mathbb{Z}\bigr\}$$

and the shift map $\sigma \colon \Sigma_A \to \Sigma_A$ defined by $\sigma(\cdots i_0 \cdots) = (\cdots j_0 \cdots)$, where $j_n = i_{n+1}$ for each $n \in \mathbb{Z}$. Finally, we define a *coding map* $\pi \colon \Sigma_A \to Z$ by

$$\pi(\cdots i_0 \cdots) = \bigcap_{n \in \mathbb{Z}} R_{i_{-n} \cdots i_n},$$

where $R_{i_{-n} \cdots i_n} = \bigcap_{l=-n}^n \overline{T_Z^{-l} \operatorname{int} R_{i_l}}$. Then the following properties hold:

(1) $\pi \circ \sigma = T \circ \pi$;
(2) π is Hölder continuous on each domain of continuity and is onto;
(3) π is one-to-one on a full measure set with respect to any ergodic measure of full support and on a residual set.

In addition, we consider the sets

$$\Sigma_A^+ = \big\{(i_0 i_1 \cdots) : (\cdots i_{-1} i_0 i_1 \cdots) \in \Sigma_A\big\}$$

and

$$\Sigma_A^- = \big\{(\cdots i_{-1} i_0) : (\cdots i_{-1} i_0 i_1 \cdots) \in \Sigma_A\big\}.$$

The shift maps $\sigma_+ \colon \Sigma_A^+ \to \Sigma_A^+$ and $\sigma_- \colon \Sigma_A^- \to \Sigma_A^-$ are defined by

$$\sigma_+(j_0 j_1 j_2 \cdots) = (j_1 j_2 \cdots) \quad \text{and} \quad \sigma_-(\cdots j_{-2} j_{-1} j_0) = (\cdots j_{-2} j_{-1}).$$

We describe briefly the relation of the symbolic dynamics to the stable and unstable manifolds. Given $x \in Z$, take $\omega \in \Sigma_A$ such that $\pi(\omega) = x$ and let $R(x)$ be the rectangle of the Markov system containing x. For each $\widetilde{\omega} \in \Sigma_A$, we have

$$\pi(\widetilde{\omega}) \in V^u(x) \cap R(x) \quad \text{whenever } \rho_+(\widetilde{\omega}) = \rho_+(\omega)$$

and

$$\pi(\widetilde{\omega}) \in V^s(x) \cap R(x) \quad \text{whenever } \rho_-(\widetilde{\omega}) = \rho_-(\omega),$$

where $\rho_+ \colon \Sigma_A \to \Sigma_A^+$ and $\rho_- \colon \Sigma_A \to \Sigma_A^-$ are the projections given by

$$\rho_+(\omega) = (i_0 i_1 \cdots) \quad \text{and} \quad \rho_-(\omega) = (\cdots i_{-1} i_0)$$

for $\omega = (\cdots i_{-1} i_0 i_1 \cdots) \in \Sigma_A$. The set $V^u(x) \cap R(x)$ is identified with the cylinder

$$C_{i_0}^+ = \big\{(j_0 j_1 \cdots) \in \Sigma_A^+ : j_0 = i_0\big\},$$

and the set $V^s(x) \cap R(x)$ is identified with the cylinder

$$C_{i_0}^- = \big\{(\cdots j_{-1} j_0) \in \Sigma_A^- : j_0 = i_0\big\}.$$

6.3. **Equilibrium measures.** Now let ν be a T_Z-invariant probability measure on Z. One can verify that ν induces a Φ-invariant probability measure μ on Λ such that

$$\int_\Lambda g\,d\mu = \frac{\int_Z \int_0^{\tau(x)} (g \circ \phi_s)(x)\,ds\,d\nu}{\int_Z \tau\,d\nu} \tag{18}$$

for any continuous function $g \colon \Lambda \to \mathbb{R}$. Moreover, any Φ-invariant probability measure μ on Λ is of this form for some T_Z-invariant probability measure ν on Z. Abramov's entropy formula says that

$$h_\mu(\Phi) = \frac{h_\nu(T_Z)}{\int_Z \tau\,d\nu}. \tag{19}$$

By (18) and (19) we have

$$h_\mu(\Phi) + \int_\Lambda g\,d\mu = \frac{h_\nu(T_Z) + \int_Z I_g\,d\nu}{\int_Z \tau\,d\nu},$$

where

$$I_g(x) = \int_0^{\tau(x)} (g \circ \phi_s)(x)\,ds.$$

6.4. **Construction of auxiliary measures.** Given pairs of families of continuous functions $(a^\pm, b^\pm) \in \mathcal{A}^\pm \times \mathcal{A}^\pm$ satisfying (9), we define sequences $c^\pm$ and $d^\pm$ by

$$c_n^\pm(x) = a^\pm_{\tau_n(x)}(x) \quad \text{and} \quad d_n^\pm(x) = b^\pm_{\tau_n(x)}(x)$$

for every $x \in Z$ and $n \in \mathbb{N}$. By Lemmas 8 and 10 in [5], the sequences $c^\pm$ and $d^\pm$ are almost additive and have bounded variation with respect to T_Z and T_Z^{-1}, respectively. Moreover, we have

$$\liminf_{n\to\infty} \frac{d_n^\pm(x)}{n} > 0 \quad \text{and} \quad d_n^\pm(x) > 0$$

for every $x \in Z$ and $n \in \mathbb{N}$. The following result is a simple adaptation of Lemma 1 in [4] for the map T_Z.

Lemma 5. *There exist sequences c^u and d^u composed of continuous functions $c_n^u, d_n^u \colon \Sigma_A^+ \to \mathbb{R}$ and numbers $\gamma_1, \gamma_2 > 0$ such that*

(1) *for every $n \in \mathbb{N}$ and $\omega \in \Sigma_A$ we have*

$$|c_n^+(\pi(\omega)) - c_n^u(\rho_+(\omega))| \le \gamma_1$$

and

$$|d_n^+(\pi(\omega)) - d_n^u(\rho_+(\omega))| \le \gamma_2;$$

(2) *c^u and d^u are almost additive sequences and have bounded variation with respect to σ_+;*

(3) *$c^+ \circ \pi$, $c^u \circ \rho_+$, $d^+ \circ \pi$ and $d^u \circ \rho_+$ are almost additive sequences and have bounded variation with respect to σ;*

(4) *$P_{T_Z}(c^+) = P_{\sigma_+}(c^u)$, $P_\sigma(c^+ \circ \pi) = P_\sigma(c^u \circ \rho_+)$, $P_{T_Z}(d^+) = P_{\sigma_+}(d^u)$ and $P_\sigma(d^+ \circ \pi) = P_\sigma(d^u \circ \rho_+)$;*

(5) *$c^+ \circ \pi$ and $c^u \circ \rho_+$ have the same equilibrium measures and $d^+ \circ \pi$ and $d^u \circ \rho_+$ have the same equilibrium measures;*

(6) *the limit*

$$\lim_{n\to\infty} \frac{(c_n^+ \circ \pi)(\omega)}{(d_n^+ \circ \pi)(\omega)}$$

exists if and only if the limit

$$\lim_{n\to\infty} \frac{(c_n^u \circ \rho_+)(\omega)}{(d_n^u \circ \rho_+)(\omega)}$$

exists, in which case they are equal.

Similarly, there also exist sequences c^s and d^s of continuous functions $c_n^s, d_n^s \colon \Sigma_A^- \to \mathbb{R}$ satisfying the statement in Lemma 5 with c^+, d^+, ρ_+, σ_+ and T_Z replaced, respectively, by c^-, d^-, ρ_-, σ_- and T_Z^{-1}.

Given $q^\pm \in \mathbb{R}$, we consider the almost additive sequences U on Σ_A^+ and S on Σ_A^- defined by

$$U = q^+(c^u - \alpha d^u) - D^+ \sum_{k=0}^{n-1} (g^u \circ \sigma_+^k)$$

and

$$S = q^-(c^s - \beta d^s) - D^- \sum_{k=0}^{n-1} (g^s \circ \sigma_-^k),$$

where

$$D^+ = \dim_H K_\alpha^+ - t_s - 1 \quad \text{and} \quad D^- = \dim_H K_\beta^- - t_u - 1, \tag{20}$$

and where

$$g^u \colon \Sigma_A^+ \to \mathbb{R} \quad \text{and} \quad g^s \colon \Sigma_A^- \to \mathbb{R}$$

are Hölder continuous functions such that $g^u \circ \rho_+$ and $g^s \circ \rho_-$ are cohomologous, respectively, to $I_{\xi_u} \circ \pi$ and $I_{-\xi_s} \circ \pi$. We note that U has bounded variation with respect to σ_+ and that S has bounded variation with respect to σ_-. Since the maps T_Z and T_Z^{-1} are topologically mixing, it follows from Theorem 12 in [1] that U has a unique equilibrium measure μ^u on Σ_A^+ (with respect to σ_+) and that S has a unique equilibrium measure μ^s on Σ_A^- (with respect to σ_-). Denoting by $\mathcal{M}_{T_Z}$ the set of all T_Z-invariant (and thus also T_Z^{-1}-invariant) probability measures on Z, we consider the maps $\mathcal{P}_{T_Z}^\pm \colon \mathcal{M}_{T_Z} \to \mathbb{R}$ defined by

$$\mathcal{P}_{T_Z}^+(\mu) := \lim_{n\to\infty} \frac{\int_Z c_n^+ \, d\mu}{\int_Z d_n^+ \, d\mu} \quad \text{and} \quad \mathcal{P}_{T_Z}^-(\mu) := \lim_{n\to\infty} \frac{\int_Z c_n^- \, d\mu}{\int_Z d_n^- \, d\mu}.$$

Lemma 6. *For each α and β as in Theorem 4, there exist $q^+, q^- \in \mathbb{R}$ with*

$$P_{\sigma_+}(U) = P_{\sigma_-}(S) = 0$$

such that the measures μ^u and μ^s satisfy

$$\lim_{n\to\infty} \frac{1}{n} \int_{\Sigma_A^+} c_n^u \, d\mu^u = \alpha \lim_{n\to\infty} \frac{1}{n} \int_{\Sigma_A^+} d_n^u d\mu^u$$

and

$$\lim_{n\to\infty} \frac{1}{n} \int_{\Sigma_A^-} c_n^s \, d\mu^s = \beta \lim_{n\to\infty} \frac{1}{n} \int_{\Sigma_A^-} d_n^s \, d\mu^s.$$

Proof. Since ξ_u is Hölder continuous, the additive family of continuous functions $\bar{\xi}_u = ((\xi_u)_t)_{t\ge 0}$ defined by

$$(\xi_u)_t(x) = \int_0^t \xi_u(\phi_s(x)) \, ds$$

has bounded variation and satisfies condition (8). Therefore, by Theorem 3.5 in [5], each linear combination of the families a^+, b^+ and $\bar{\xi}_u$ has a unique equilibrium measure. This allows us to apply Theorem 8 in [6] to conclude that for each $\alpha \in \operatorname{int} \mathcal{P}_\Phi^+(\mathcal{M}_\Phi)$ there exists an ergodic measure $\mu_\alpha \in \mathcal{M}_\Phi$ such that

$$\alpha = \mathcal{P}_\Phi^+(\mu_\alpha) = \lim_{t\to\infty} \frac{\int_\Lambda a_t^+ \, d\mu_\alpha}{\int_\Lambda b_t^+ \, d\mu_\alpha}.$$

Moreover, by Lemma 3.4 in [5] we have

$$\lim_{t\to\infty} \frac{1}{t} \int_\Lambda a_t^\pm \, d\mu_\alpha = \lim_{n\to\infty} \frac{1}{n} \int_Z c_n^\pm \, d\nu_\alpha \Big/ \int_Z \tau \, d\nu,$$

where ν_α is the T_Z-invariant measure on Z that induces the measure μ_α on Λ as in (18), and so

$$\lim_{t\to\infty} \frac{\int_\Lambda a_t^+ \, d\mu_\alpha}{\int_\Lambda b_t^+ \, d\mu_\alpha} = \lim_{n\to\infty} \frac{\int_Z c_n^+ \, d\nu_\alpha}{\int_Z d_n^+ \, d\nu_\alpha} = \mathcal{P}_{T_Z}^+(\nu_\alpha).$$

Therefore, $\alpha \in \operatorname{int} \mathcal{P}_{T_Z}(\mathcal{M}_{T_Z})$. By Lemma 5, for each $\nu \in \mathcal{M}_{T_Z}$ we have

$$\begin{aligned}\mathcal{P}^+_{T_Z}(\nu) &= \lim_{n\to\infty} \frac{\int_{\Sigma_A} c_n^+ \circ \pi \, dm}{\int_{\Sigma_A} d_n^+ \circ \pi \, dm} \\ &= \lim_{n\to\infty} \frac{\int_{\Sigma_A} c_n^u \circ \rho_+ \, dm}{\int_{\Sigma_A} d_n^u \circ \rho_+ \, dm} = \lim_{n\to\infty} \frac{\int_{\Sigma_A^+} c_n^u \, d\eta}{\int_{\Sigma_A^+} d_n^u \, d\eta},\end{aligned}$$

where $m = \nu \circ \pi$ and $\eta = m \circ \rho_+^{-1}$. Therefore, denoting by $\mathcal{M}_{\sigma_\pm}$ the set of all $\sigma_\pm$-invariant probability measures on $\Sigma_A^\pm$ and letting

$$\mathcal{P}^+_{\sigma_+}(\eta) = \lim_{n\to\infty} \frac{\int_{\Sigma_A^+} c_n^u \, d\eta}{\int_{\Sigma_A^+} d_n^u \, d\eta} \quad \text{and} \quad \mathcal{P}^-_{\sigma_-}(\eta) = \lim_{n\to\infty} \frac{\int_{\Sigma_A^-} c_n^s \, d\eta}{\int_{\Sigma_A^-} d_n^s \, d\eta},$$

we conclude that $\alpha \in \operatorname{int} \mathcal{P}^+_{\sigma_+}(\mathcal{M}_{\sigma_+})$. Hence, it follows from Theorem 3 in [3] that there exists $q^+(\alpha) \in \mathbb{R}$ such that

$$P_{\sigma_+}(U) = 0 \quad \text{and} \quad \mathcal{P}^+_{\sigma_+}(\mu^u) = \alpha.$$

One can show in a similar manner that there exists $q^-(\beta) \in \mathbb{R}$ such that

$$P_{\sigma_-}(S) = 0 \quad \text{and} \quad \mathcal{P}^-_{\sigma_-}(\mu^s) = \beta.$$

This completes the proof of the lemma. □

6.5. **Estimates for the pointwise dimension.** Recall that $Z = \bigcup_{i=1}^k R_i$. We denote by $R(x)$ the rectangle of the Markov system that contains x. Taking q^+ and q^- as in Lemma 6 (notice that μ^u depends on q^+ and that μ^s depends on q^-), we define measures ν^u and ν^s on $R(x)$ by

$$\nu^u = \mu^u \circ \rho_+ \circ \pi^{-1} \quad \text{and} \quad \nu^s = \mu^s \circ \rho_- \circ \pi^{-1}.$$

We also define a measure ν on $R(x)$ by $\nu = \nu^u \times \nu^s$. Since the measures μ^u and μ^s have the Gibbs property (see for example Theorem 11 in [5]), for $x = \pi(\omega)$ with $\omega = (\cdots i_0 \cdots)$ we have

$$\nu(R(x)) = \mu^u(C^+_{i_0})\mu^s(C^-_{i_0}) > 0.$$

Lemma 7. *For ν-almost every $x \in Z$, we have*

$$\liminf_{r\to 0} \frac{\log \nu(B(x,r))}{\log r} \geq \dim_H K^+_\alpha + \dim_H K^+_\beta - \dim_H \Lambda - 1.$$

Proof. It follows from Lemma 6 and Birkhoff's ergodic theorem that

$$\begin{aligned}0 &= P_{\sigma_+}(U) \\ &= h_{\mu^u}(\sigma_+) + \lim_{n\to\infty} \frac{1}{n} \int_{\Sigma_A^+} U_n \, d\mu^u \\ &= h_{\mu^u}(\sigma_+) - D^+ \frac{1}{n} \int_{\Sigma_A^+} \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} (g^u \circ \sigma_+^k) \, d\mu^u \\ &= h_{\mu^u}(\sigma_+) - D^+ \int_{\Sigma_A^+} g^u \, d\mu^u.\end{aligned}$$

Similarly, we also obtain

$$0 = P_{\sigma_-}(S) = h_{\mu^s}(\sigma_-) - D^- \int_{\Sigma_A^-} g^s \, d\mu^s.$$

This implies that

$$D^+ = \frac{h_{\mu^u}(\sigma_+)}{\int_{\Sigma_A^+} g^u \, d\mu^u} \quad \text{and} \quad D^- = \frac{h_{\mu^u}(\sigma_-)}{\int_{\Sigma_A^-} g^s \, d\mu^s}. \tag{21}$$

By the Shannon–McMillan–Breiman theorem, we have

$$\begin{aligned} h_{\mu^u}(\sigma_+) &= \int_{\Sigma_A^+} \lim_{n\to\infty} -\frac{1}{n} \log \mu^u(C^+_{i_0\cdots i_n}) \, d\mu^u(\omega_+), \\ h_{\mu^s}(\sigma_-) &= \int_{\Sigma_A^-} \lim_{m\to\infty} -\frac{1}{m} \log \mu^s(C^-_{i_{-m}\cdots i_0}) \, d\mu^s(\omega_-). \end{aligned} \tag{22}$$

Moreover, since μ^u and μ^s are ergodic measures, it follows from Birkhoff's ergodic theorem that

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^n (g^u \circ \sigma_+^k)(\omega_+) &= \int_{\Sigma_A^+} g^u \, d\mu^u, \\ \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^n (g^s \circ \sigma_-^k)(\omega_-) &= \int_{\Sigma_A^-} g^s \, d\mu^s \end{aligned} \tag{23}$$

for μ^u-almost every $\omega_+ \in \Sigma_A^+$ and μ^s-almost every $\omega_- \in \Sigma_A^-$.

By (21) together with (22) and (23), given $\varepsilon > 0$, for μ^u-almost every $\rho_+(\omega) = \omega_+ \in C^+_{i_0}$ and μ^s-almost every $\rho_-(\omega) = \omega_- \in C^-_{i_0}$, there exists $N_1(\omega) \in \mathbb{N}$ such that

$$\begin{aligned} D^+ - \varepsilon &< -\frac{\log \mu^u(C^+_{i_0\cdots i_n})}{\sum_{k=0}^n (g^u \circ \sigma_+^k)(\omega_+)} < D^+ + \varepsilon, \\ D^- - \varepsilon &< -\frac{\log \mu^s(C^-_{i_{-m}\cdots i_0})}{\sum_{k=0}^m (g^u \circ \sigma_-^k)(\omega_-)} < D^- + \varepsilon \end{aligned} \tag{24}$$

for all $n, m > N_1(\omega)$. On the other hand, since $\inf \tau > 0$, we have

$$\begin{aligned} I_{\xi_u}(\pi(\omega)) &= \int_0^{\tau(\pi(\omega))} (\xi_u \circ \phi_q)(\pi(\omega)) \, dq \ge \inf \tau(-\log \lambda) > 0, \\ I_{-\xi_s}(\pi(\omega)) &= \int_0^{\tau(\pi(\omega))} (-\xi_s \circ \phi_q)(\pi(\omega)) \, dq \ge \inf \tau(-\log \lambda) > 0 \end{aligned} \tag{25}$$

for all $\omega \in \Sigma_A$. Since $g^u \circ \rho_+$ is cohomologous to $I_{\xi_u} \circ \pi$ and $g^s \circ \rho_-$ is cohomologous to $I_{-\xi_s} \circ \pi$, there exist bounded measurable functions $\psi^u, \psi^s \colon \Sigma_A \to \mathbb{R}$ such that

$$g^u \circ \rho_+ - I_{\xi_u} \circ \pi = \psi^u \circ \sigma_+ - \psi^u$$

and

$$g^s \circ \rho_- - I_{-\xi_s} \circ \pi = \psi^s \circ \sigma_- - \psi^s.$$

Since ψ^s and ψ^u are bounded, it follows from (25) that for any sufficiently small $r > 0$ there exist unique integers $n = n(\omega, r)$ and $m = m(\omega, r)$ with

$$\begin{aligned} \sum_{k=0}^n (g^u \circ \sigma_+^k)(\omega_+) &< -\log r \le \sum_{k=0}^{n+1} (g^u \circ \sigma_+^k)(\omega_+), \\ \sum_{k=0}^m (g^s \circ \sigma_-^k)(\omega_-) &< -\log r \le \sum_{k=0}^{m+1} (g^u \circ \sigma_+^k)(\omega_-). \end{aligned} \tag{26}$$

Moreover, we have the following result.

Lemma 8. $r \to 0$ *if and only if* $n(\omega, r), m(\omega, r) \to \infty$.

Proof of the lemma. It follows from (26) that

$$\begin{aligned} -\log r &\le \sum_{k=0}^{n+1} (g^u \circ \sigma_+^k)(\omega_+) \\ &= \sum_{k=0}^{n+1} (I_{\xi_u} \circ \pi)(\sigma^k(\omega)) + \psi^u(\sigma_+^{n+2}(\omega)) - \psi^u(\omega) \\ &\le (n+2)\|I_{\xi_u}\|_\infty + 2\|\psi^u\|_\infty \end{aligned} \tag{27}$$

and

$$\begin{aligned} -\log r &> \sum_{k=0}^{n} (g^u \circ \sigma_+^k)(\omega_+) \\ &= \sum_{k=0}^{n} (I_{\xi_u} \circ \pi)(\sigma^k(\omega)) + \psi^u(\sigma_+^{n+1}(\omega)) - \psi^u(\omega) \\ &\ge (n+1) \inf I_{\xi_u} - 2\|\psi^u\|_\infty. \end{aligned} \tag{28}$$

Analogously, we also have

$$-\log r \le (m+2)\|I_{-\xi_s}\|_\infty + 2\|\psi^s\|_\infty \tag{29}$$

and

$$-\log r > (m+1) \inf I_{-\xi_s} - 2\|\psi^s\|_\infty. \tag{30}$$

It follows readily from (25) together with (27), (28), (29) and (30) that $r \to 0$ if and only if $n(\omega, r), m(\omega, r) \to \infty$. ☐

Proceeding as in the proof of Lemma 3 in [4], we find that given $\varepsilon > 0$, there exist $\varrho > 1$ (independent of r and $x = \pi(\omega)$) and $\delta = \delta(x, \varepsilon, \varrho) > 0$ such that

$$\begin{aligned} \nu(B(x,r)) &\le \nu(C_{i_{-m}\cdots i_n}) \left(\frac{r}{\varrho}\right)^{-\varepsilon} \\ &= \mu^u(C^+_{i_0\cdots i_n})\mu^s(C^-_{i_{-m}\cdots i_0}) \left(\frac{r}{\varrho}\right)^{-\varepsilon} \end{aligned}$$

for $r < \delta$ and for ν-almost every $x \in Z$. By Lemma 8 together with (24) and (26), we obtain

$$\nu(B(x,r)) \le \exp\big[(\log r + \|g^u\|_\infty)(D^+ - \varepsilon) + (\log r + \|g^s\|_\infty)(D^- - \varepsilon)\big] \left(\frac{r}{\varrho}\right)^{-\varepsilon}$$

and taking $r < 1$ sufficiently small yields the inequality

$$\begin{aligned} \frac{\log \nu(B(x,r))}{\log r} &\ge D^+ + D^- - 3\varepsilon + \frac{\|g^u\|_\infty (D^+ - \varepsilon)}{\log r} \\ &\quad + \frac{\|g^s\|_\infty (D^- - \varepsilon)}{\log r} + \frac{\varepsilon \log \varrho}{\log r}. \end{aligned}$$

Finally, since $\varepsilon > 0$ is arbitrary, we conclude that

$$\liminf_{r \to 0} \frac{\log \nu(B(x,r))}{\log r} \ge D^+ + D^- \tag{31}$$

for ν-almost every $x \in Z$. On the other hand, by Theorem 4.2 in [12] we have

$$\dim_H \Lambda = t_u + t_s + 1 \tag{32}$$

and it follows from (20) and (31) that

$$\begin{aligned}\liminf_{r\to 0} \frac{\log \nu(B(x,r))}{\log r} &\ge \dim_H K_\alpha^+ + \dim_H K_\beta^- - (t_u + t_s + 1) - 1\\ &= \dim_H K_\alpha^+ + \dim_H K_\beta^- - \dim_H \Lambda - 1.\end{aligned}$$

This completes the proof of the lemma. □

Lemma 9. *For each $x \in K_\alpha^+ \cap K_\beta^- \cap Z$, we have*

$$\limsup_{r\to 0} \frac{\log \nu(B(x,r))}{\log r} \le \dim_H K_\alpha^+ + \dim_H K_\beta^+ - \dim_H \Lambda - 1.$$

Proof. By Lemma 12 in [6], we have

$$\limsup_{t\to\infty} \frac{a_t^\pm(x)}{b_t^\pm(x)} = \limsup_{n\to\infty} \frac{c_n^\pm(x)}{d_n^\pm(x)} \quad \text{and} \quad \liminf_{t\to\infty} \frac{a_t^\pm(x)}{b_t^\pm(x)} = \liminf_{n\to\infty} \frac{c_n^\pm(x)}{d_n^\pm(x)}$$

for every $x \in Z$. Therefore,

$$\begin{aligned}K_\alpha^+ \cap K_\beta^- \cap Z &= \left\{x \in Z : \lim_{t\to\infty} \frac{a_t^+(x)}{b_t^+(x)} = \alpha \text{ and } \lim_{t\to\infty} \frac{a_t^-(x)}{b_t^-(x)} = \beta\right\}\\ &= \left\{x \in Z : \lim_{n\to\infty} \frac{c_n^+(x)}{d_n^+(x)} = \alpha \text{ and } \lim_{n\to\infty} \frac{c_n^-(x)}{d_n^-(x)} = \beta\right\}.\end{aligned}$$

Now take $x \in K_\alpha^+ \cap K_\beta^- \cap Z$ and $\omega \in \Sigma_A$ such that $\pi(\omega) = x$, with projections $\omega_+ = \rho_+(\omega)$ and $\omega_- = \rho_-(\omega)$. It follows from item (6) in Lemma 5 that

$$\begin{aligned}\lim_{n\to\infty} \frac{c_n^u(\omega_+)}{d_n^u(\omega_+)} &= \lim_{n\to\infty} \frac{(c_n^u \circ \rho_+)(\omega)}{(d_n^u \circ \rho_+)(\omega)}\\ &= \lim_{n\to\infty} \frac{(c_n^+ \circ \pi)(\omega)}{(d_n^+ \circ \pi)(\omega)}\\ &= \lim_{n\to\infty} \frac{c_n^+(x)}{d_n^+(x)} = \alpha\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty} \frac{c_n^s(\omega_-)}{d_n^s(\omega_-)} &= \lim_{n\to\infty} \frac{(c_n^s \circ \rho_-)(\omega)}{(d_n^s \circ \rho_-)(\omega)}\\ &= \lim_{n\to\infty} \frac{(c_n^- \circ \pi)(\omega)}{(d_n^- \circ \pi)(\omega)}\\ &= \lim_{n\to\infty} \frac{c_n^-(x)}{d_n^-(x)} = \beta.\end{aligned}$$

Hence, given $\varepsilon > 0$, there exists $N_2(\omega) \in \mathbb{N}$ such that

$$|c_n^u(\omega_+) - \alpha d_n^u(\omega_+)| \le d_n^u(\omega_+)\varepsilon \le n\varepsilon(\|d_1^u\|_\infty + C_1) \tag{33}$$

and

$$|c_m^s(\omega_-) - \beta d_m^u(\omega_-)| \le d_m^s(\omega_-)\varepsilon \le m\varepsilon(\|d_1^s\|_\infty + C_2) \tag{34}$$

for every $n, m > N_2(\omega)$, where $C_1, C_2 > 0$ are the constants in the notion of almost additivity. By (33), for $n > N_2(\omega)$ we have

$$\begin{aligned} U_n(\omega_+) &\geq -|q^+| \cdot |c_n^u(\omega_+) - \alpha d_n^u(\omega_+)| - D^+ \sum_{k=0}^{n} (g^u \circ \sigma_+^k)(\omega_+) \\ &\geq -|q^+|(\|d_1^u\|_\infty + C_1)n\varepsilon - D^+ \sum_{k=0}^{n} (g^u \circ \sigma_+^k)(\omega_+). \end{aligned}$$

Similarly, by (34), for $m > N_2(\omega)$ we have

$$S_m(\omega_-) \geq -|q^-|(\|d_1^s\|_\infty + C_2)m\varepsilon - D^- \sum_{k=0}^{m} (g^s \circ \sigma_-^k)(\omega_-).$$

On the other hand, since the measures μ^u and μ^s have the Gibbs property and $P_{\sigma_+}(U) = P_{\sigma_-}(S) = 0$ (by Lemma 6), there exist constants $C_3, C_4 > 0$ such that

$$\frac{1}{C_3} \leq \frac{\mu^u(C^+_{i_0 \cdots i_n})}{\exp U_n(\omega_+)} \leq C_3 \quad \text{and} \quad \frac{1}{C_4} \leq \frac{\mu^s(C^-_{i_{-m} \cdots i_0})}{\exp S_m(\omega_-)} \leq C_4$$

for every $\omega = (\cdots i_{-1} i_0 i_1 \cdots) \in \Sigma_A$ and $n, m \in \mathbb{N}$. For $n, m > N_2(\omega)$ we obtain

$$\mu^u(C^+_{i_0 \cdots i_n}) \geq \frac{1}{C_3} \exp\left[-|q^+|(\|d_1^u\|_\infty + C_1)n\varepsilon - D^+ \sum_{k=0}^{n} (g^u \circ \sigma_+^k)(\omega_+) \right] \quad (35)$$

and

$$\mu^s(C^-_{i_{-m} \cdots i_0}) \geq \frac{1}{C_4} \exp\left[-|q^-|(\|d_1^s\|_\infty + C_2)m\varepsilon - D^- \sum_{k=0}^{m} (g^s \circ \sigma_-^k)(\omega_-) \right]. \quad (36)$$

Lemma 8 implies that for any sufficiently small $r = r(\omega) > 0$ we have

$$n(\omega, r) > \max\{N_1(\omega), N_2(\omega)\} \quad \text{and} \quad m(\omega, r) > \max\{N_1(\omega), N_2(\omega)\}.$$

Proceeding as in the proof of Lemma 4 in [4], we find that there exists $\varsigma > 0$ (independent of r and $x = \pi(\omega)$) such that

$$\nu(B(x, \varsigma r)) \geq \nu(\pi(C_{i_{-m} \cdots i_n})) = \mu^u(C^+_{i_0 \cdots i_n}) \mu^s(C^-_{i_{-m} \cdots i_0})$$

with $m = m(\omega, r)$ and $n = n(\omega, r)$. By (26), (35) and (36), we obtain

$$\nu(B(x, \varsigma r)) \geq \frac{r^{D^+} r^{D^-}}{C_3 C_4} \exp\left[-|q^+|(\|d_1^u\|_\infty + C_1)n\varepsilon - |q^-|(\|d_1^s\|_\infty + C_2)m\varepsilon \right] \quad (37)$$

for any sufficiently small $r > 0$. On the other hand, it follows from (28) and (30) that

$$-\frac{n}{\log r} < \frac{1}{\inf I_{\xi_u}} - \frac{2\|\psi^u\|_\infty}{\inf I_{\xi_u} \log r}$$

and

$$-\frac{m}{\log r} < \frac{1}{\inf I_{-\xi_s}} - \frac{2\|\psi^s\|_\infty}{\inf I_{-\xi_s} \log r}.$$

Therefore, by (37) we obtain

$$\begin{aligned}\frac{\log \nu(B(x,\varsigma r))}{\log r} &\le D^+ + D^- - \frac{\log(C_3C_4)}{\log r}\\ &\quad + \varepsilon|q^+|(\|d_1^u\|_\infty + C_1)\left(\frac{1}{\inf I_{\xi_u}} - \frac{2\|\psi^u\|_\infty}{\inf I_{\xi_u}\log r}\right)\\ &\quad + \varepsilon|q^-|(\|d_1^s\|_\infty + C_2)\left(\frac{1}{\inf I_{-\xi_s}} - \frac{2\|\psi^s\|_\infty}{\inf I_{-\xi_s}\log r}\right)\end{aligned}$$

for any sufficiently small $r > 0$. Since $\varepsilon > 0$ is arbitrary, we conclude that

$$\limsup_{r\to 0} \frac{\nu(B(x,r))}{\log r} \le D^+ + D^-.$$

The desired result follows now readily from (20) and (32). □

6.6. **Conclusion of the proof.**

Proof of Theorem 4. For each α and β as in the statement of the theorem, we constructed a measure ν on Z with $\nu(K_\alpha^+ \cap K_\beta^-) = 1$. By Lemmas 7 and 9, it follows for example from Theorem 2.1.5 in [2] that

$$\dim_H(K_\alpha^+ \cap K_\beta^- \cap Z) = \dim_H K_\alpha^+ + \dim_H K_\beta^- - \dim_H \Lambda - 1.$$

Since $K_\alpha^+ \cap K_\beta^- \subset \Lambda$ is locally diffeomorphic to $(K_\alpha^+ \cap K_\beta^- \cap Z) \times I$, where $I \subset \mathbb{R}$ is some open interval, we obtain

$$\mathcal{D}(\alpha,\beta) = \dim_H(K_\alpha^+ \cap K_\beta^-) = \dim_H K_\alpha^+ + \dim_H K_\beta^- - \dim_H \Lambda. \tag{38}$$

Applying the identities in (14) and (15), we find that

$$\mathcal{D}(\alpha,\beta) = \dim_{\xi_u} K_\alpha^+ + \dim_{-\xi_s} K_\beta^- + 1. \tag{39}$$

Now consider the additive families $\bar{\xi}_u = (\xi_u)_{t\ge 0}$ and $\bar{\xi}_s = (-\xi_s)_{t\ge 0}$ given by

$$(\xi_u)_t(x) = \int_0^t (\xi_u \circ \phi_q)(x)\,dq \quad\text{and}\quad (-\xi_s)_t(x) = \int_0^t (-\xi_s \circ \phi_q)(x)\,dq.$$

By Theorem 3.5 in [5], any family of functions that is a linear combination of $a^\pm$, $b^\pm$, $\bar{\xi}_u$ and $\bar{\xi}_s$ has a unique equilibrium measure. Hence, it follows from Theorem 8 in [6] that

$$\dim_{\xi_u} K_\alpha^+ = \max\left\{\frac{h_\mu(\Phi)}{\int_\Lambda \xi_u\,d\mu} : \mu \in \mathcal{M}_\Phi \text{ and } \mathcal{P}_\Phi^+(\mu) = \alpha\right\}$$

and

$$\dim_{-\xi_s} K_\beta^- = \max\left\{\frac{h_\mu(\Phi)}{\int_\Lambda -\xi_s\,d\mu} : \mu \in \mathcal{M}_\Phi \text{ and } \mathcal{P}_\Phi^-(\mu) = \beta\right\}.$$

The first statement in the theorem follows now readily from (38) and (39). To prove the second statement, note that by Theorem 8 in [6] we also have

$$\dim_{\xi_u} K_\alpha^+ = \min\{S_{\xi_u}(\alpha,q) : q \in \mathbb{R}\},$$

where $S_{\xi_u}(\alpha,q)$ is the unique real number such that

$$P_\Phi\big(q(a^+ - \alpha b^+) - S_{\xi_u}(\alpha,q)\bar{\xi}_u\big) = 0.$$

By Proposition 5 in [6], the function

$$(q,\alpha,p) \mapsto P_\Phi\big(q(a^+ - \alpha b^+) - p\bar{\xi}_u\big)$$

is of class C^1. Hence, applying the implicit function theorem, we find that $(\alpha, q) \mapsto S_{\xi_u}(\alpha, q)$ is also of class C^1, which implies that the map

$$\operatorname{int} \mathcal{P}_\Phi^+(\mathcal{M}_\Phi) \ni \alpha \mapsto \dim_{\xi_u} K_\alpha^+$$

is continuous. One can show in a similar manner that the map

$$\operatorname{int} \mathcal{P}_\Phi^-(\mathcal{M}_\Phi) \ni \beta \mapsto \dim_{-\xi_s} K_\beta^-$$

is continuous, which completes the proof of the theorem. □

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