EQUILIBRIUM AND GIBBS MEASURES FOR FLOWS

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Dedicated to Yakov Sinai on the occasion of his 85th birthday

Abstract. We construct equilibrium and Gibbs measures in the context of the nonadditive thermodynamic formalism for flows. More precisely, we consider the class of almost additive families of potentials and after establishing an appropriate version of the classical variational principle for the topological pressure, we obtain the existence and uniqueness of equilibrium and Gibbs measures for families with bounded variation.

1. Introduction

We first recall some of the main components of the classical thermodynamic formalism. The notion of the topological pressure $P(\phi)$ of a continuous function ϕ with respect to a map $f\colon X \to X$ was introduced by Ruelle in [15] for expansive maps and by Walters in [17] in the general case. They also established a variational principle for the topological pressure:

$$P(\phi) = \sup_\mu \left(h_\mu(f) + \int_X \phi \, d\mu \right),$$

with the supremum taken over all f-invariant probability measures μ on X, denoting by $h_\mu(f)$ the Kolmogorov–Sinai entropy of f with respect to μ. An f-invariant probability measure μ on X is called an *equilibrium measure* for ϕ if

$$P(\phi) = h_\mu(f) + \int_X \phi \, d\mu.$$

These measures and particularly their Gibbs property play an important role in the dimension theory and multifractal analysis of dynamical systems. We refer the reader to the books [3, 6, 12, 16] for details and further references.

The nonadditive thermodynamic formalism was introduced in [1] as a generalization of the classical thermodynamic formalism, essentially replacing the topological pressure $P(\phi)$ by the topological pressure $P(\Phi)$ of a sequence of continuous functions $\Phi = (\phi_n)_{n\in\mathbb{N}}$. This formalism contains as a particular case a new formulation of the subadditive thermodynamic formalism introduced by Falconer in [8]. Moreover, for additive sequences it recovers the notion of topological pressure introduced by Pesin and Pitskel' in [13] as well as the notions of lower and upper capacity topological pressures introduced by Pesin in [11] for an arbitrary set. The nonadditive thermodynamic

2010 *Mathematics Subject Classification.* Primary: 28D20, 37D35.

Key words and phrases. Thermodynamic formalism, topological pressure, flows.

Partially supported by FCT/Portugal through UID/MAT/04459/2013. C.H. was supported by FCT/Portugal through the grant PD/BD/135523/2018.

formalism also plays a corresponding role in the dimension theory of dynamical systems. In particular, [1] includes a version of the variational principle for the topological pressure (for discrete time), although with restrictive assumptions on the sequence Φ. This justifies the interest in looking for more general classes of sequences of functions for which it is still possible to establish a variational principle, including in the case of flows.

Our main objective is precisely to consider a new class of families for which it is still possible not only to establish a variational principle for the topological pressure, but also to discuss the existence and uniqueness of equilibrium and Gibbs measures. This is the class of almost additive families: a family of functions $(a_t)_{t\geq 0}$ is said to be *almost additive* with respect to a flow $(\phi_t)_{t\in\mathbb{R}}$ if there exists a constant $C>0$ such that

$$-C + a_t + a_s \circ \phi_t \leq a_{t+s} \leq a_t + a_s \circ \phi_t + C$$

for every $t, s \geq 0$. In particular, we establish the following variational principle for the topological pressure. We denote by $\mathcal{M}$ the set of all Φ-invariant probability measures on X and we refer to Section 2 for the notion of tempered variation.

Theorem 1. *Let Φ be a continuous flow on a compact metric space X and let a be an almost additive family of continuous functions with tempered variation such that $\sup_{t\in[0,s]}\|a_t\|_\infty < \infty$ for some $s>0$. Then*

$$P(a) = \sup_{\mu\in\mathcal{M}} \left(h_\mu(\Phi) + \lim_{t\to\infty} \frac{1}{t} \int_X a_t \, d\mu \right). \tag{1}$$

We also consider the particular case of hyperbolic flows and we establish the existence and uniqueness of the equilibrium measure of an almost additive family of continuous functions with bounded variation as well as its Gibbs property. We say that a Φ-invariant measure μ on X is an *equilibrium measure* for the almost additive family a (with respect to the flow Φ) if the supremum in (1) is attained at μ, that is, if

$$P(a) = h_\mu(\Phi) + \lim_{t\to\infty} \frac{1}{t} \int_X a_t \, d\mu.$$

The notion of a Gibbs measure requires introducing the somewhat technical notion of a Markov system (see Section 3.1). Our main result is the following theorem.

Theorem 2. *Let Λ be a hyperbolic set for a topologically mixing C^1 flow Φ and let a be an almost additive family of continuous functions on Λ with bounded variation such that $P(a)=0$ and $\sup_{t\in[0,s]}\|a_t\|_\infty < \infty$ for some $s>0$. Then:*

1. *there exists a unique equilibrium measure for a;*
2. *there exists a unique invariant Gibbs measure for a;*
3. *the two measures are equal and are mixing.*

Note that there is no loss of generality in assuming that $P(a)=0$. Indeed, let $b=(b_t)_{t\geq 0}$ be an almost additive family of continuous functions on Λ with bounded variation such that $\sup_{t\in[0,s]}\|b_t\|_\infty < \infty$ for some $s>0$. Then let $a=(a_t)_{t\geq 0}$ be the family of continuous functions on Λ defined by

$$a_t = b_t - P(b)t$$

for each $t \geq 0$. Clearly, a is almost additive, has bounded variation and satisfies $\sup_{t\in[0,s]} \|a_t\|_\infty < \infty$ and $P(a) = 0$. For each Φ-invariant probability measure μ on Λ we have

$$\lim_{t\to\infty} \frac{1}{t} \int_X a_t \, d\mu = \lim_{t\to\infty} \frac{1}{t} \int_X b_t \, d\mu - P(b).$$

This readily implies that a and b have the same equilibrium measures.

To the possible extent, and up to the need of various nontrivial modifications in the case of flows, our arguments follow former work in [2] for discrete time.

2. Variational principle

In this section we consider the nonadditive topological pressure for a flow and we establish a version of the variational principle for an almost additive family of continuous functions.

We first recall the notion of nonadditive topological pressure for a flow. Let $\Phi = (\phi_t)_{t\in\mathbb{R}}$ be a continuous flow on a compact metric space (X, d). Moreover, let $a = (a_t)_{t\geq 0}$ be a family of continuous functions $a_t \colon X \to \mathbb{R}$ with tempered variation. This means that

$$\lim_{\varepsilon\to 0} \overline{\lim_{t\to+\infty}} \frac{\gamma_t(a,\varepsilon)}{t} = 0,$$

where

$$\gamma_t(a,\varepsilon) = \sup\bigl\{|a_t(y) - a_t(x)| : y \in B_t(x,\varepsilon) \text{ for some } x \in X\bigr\}$$

taking

$$B_t(x,\varepsilon) = \bigl\{y \in X : d(\phi_s(y), \phi_s(x)) < \varepsilon \text{ for } s \in [0,t]\bigr\}. \tag{2}$$

Given $\varepsilon > 0$, we say that a set $\Gamma \subset X \times \mathbb{R}_0^+$ *covers* $Z \subset X$ if

$$\bigcup_{(x,t)\in\Gamma} B_t(x,\varepsilon) \supset Z$$

and we write

$$a(x,t,\varepsilon) = \sup\bigl\{a_t(y) : y \in B_t(x,\varepsilon)\bigr\} \quad \text{for } (x,t) \in \Gamma.$$

For each $Z \subset X$ and $\alpha \in \mathbb{R}$, let

$$M(Z,a,\alpha,\varepsilon) = \lim_{T\to+\infty} \inf_{\Gamma} \sum_{(x,t)\in\Gamma} \exp(a(x,t,\varepsilon) - \alpha t), \tag{3}$$

with the infimum taken over all countable sets $\Gamma \subset X \times [T,+\infty)$ covering Z. When α goes from $-\infty$ to $+\infty$, the quantity in (3) jumps from $+\infty$ to 0 at a unique value and so one can define

$$P(a|_Z, \varepsilon) = \inf\bigl\{\alpha \in \mathbb{R} : M(Z,a,\alpha,\varepsilon) = 0\bigr\}.$$

Moreover, the limit

$$P(a|_Z) = \lim_{\varepsilon\to 0} P(a|_Z, \varepsilon)$$

exists and is called the *nonadditive topological pressure* of the family a on the set Z. For simplicity of the notation, we shall also write $P(a|_X) = P(a)$.

Now we establish a version of the variational principle for the topological pressure of an almost additive family of continuous functions. We recall that a family $a = (a_t)_{t\ge 0}$ of functions $a_t \colon X \to \mathbb{R}$ is said to be *almost additive* (with respect to a flow Φ) if there exists a constant $C > 0$ such that

$$-C + a_t + a_s \circ \phi_t \le a_{t+s} \le a_t + a_s \circ \phi_t + C$$

for every $t, s \ge 0$. We denote by $\mathcal{M}$ the set of all Φ-invariant probability measures on X, that is, the probability measures μ on X such that

$$\mu(\phi_t(A)) = \mu(A)$$

for any Borel set $A \subset X$ and any $t \in \mathbb{R}$. Moreover, for each $\mu \in \mathcal{M}$, let $h_\mu(\Phi)$ be the Kolmogorov–Sinai entropy of Φ with respect to μ.

Theorem 3. *Let Φ be a continuous flow on a compact metric space X and let a be an almost additive family of continuous functions with tempered variation such that* $\sup_{t\in[0,s]} \|a_t\|_\infty < \infty$ *for some $s > 0$. Then*

$$\begin{aligned} P(a) &= \sup_{\mu\in\mathcal{M}} \left(h_\mu(\Phi) + \int_X \lim_{t\to\infty} \frac{a_t(x)}{t}\, d\mu(x) \right) \\ &= \sup_{\mu\in\mathcal{M}} \left(h_\mu(\Phi) + \lim_{t\to\infty} \frac{1}{t} \int_X a_t \, d\mu \right). \end{aligned} \tag{4}$$

Proof. Since a is almost additive, we have

$$a_{t+s} + C \le (a_t + C) + a_s \circ \phi_t + C$$

for $s, t \ge 0$. Thus, $(a_n + C)_{n\in\mathbb{N}}$ is subadditive and it follows from Kingman's subadditive ergodic theorem that for each measure $\mu \in \mathcal{M}$ the limit

$$\tilde{a}(x) = \lim_{n\to\infty} (a_n(x)/n)$$

exists for μ-almost every $x \in X$. Now let $[x]$ be the integer part of the real number x. Again since a is almost additive, we have

$$-C + a_{[t]} + a_{t-[t]} \circ \phi_{[t]} \le a_t \le a_{[t]} + a_{t-[t]} \circ \phi_{[t]} + C \tag{5}$$

for $t > 0$. Taking $N \in \mathbb{N}$ such that $1/N < s$ (with s as in the statement of the theorem), we obtain

$$\begin{aligned} \left| \frac{a_t(x)}{t} - \frac{a_{[t]}(x)}{t} \right| &\le \left| \frac{(a_{t-[t]} \circ \phi_{[t]})(x)}{t} \right| + \frac{C}{t} \\ &\le \frac{\sup_{t\in[0,1]} \|a_t\|_\infty}{t} + \frac{C}{t} \\ &\le \frac{N \sup_{t\in[0,1/N]} \|a_t\|_\infty}{t} + \frac{NC}{t} \\ &\le \frac{N \sup_{t\in[0,s]} \|a_t\|_\infty}{t} + \frac{NC}{t}. \end{aligned}$$

Taking the limit when $t \to \infty$ gives

$$\lim_{t\to\infty} \left| \frac{a_t(x)}{t} - \frac{a_{[t]}(x)}{t} \right| = 0. \tag{6}$$

Since

$$\lim_{t\to\infty} \frac{a_{[t]}(x)}{t} = \lim_{t\to\infty} \frac{[t]}{t} \frac{a_{[t]}(x)}{[t]} = \lim_{t\to\infty} \frac{a_{[t]}(x)}{[t]} = \tilde{a}(x),$$

it follows from (6) that

$$\lim_{t\to\infty}\frac{a_t(x)}{t}=\tilde a(x)$$

for μ-almost every $x\in X$. Moreover,

$$-C[t]+\sum_{k=0}^{[t]-1}a_1\circ\phi_k\le a_{[t]}\le\sum_{k=0}^{[t]-1}a_1\circ\phi_k+C[t]$$

and so $|a_{[t]}/[t]|\le\|a_1\|_\infty+C$. Hence, it follows from Lebesgue's dominated convergence theorem that $a_{[t]}/[t]\to\tilde a$ in $L^1(X,\mu)$ when $t\to\infty$ and

$$\lim_{t\to\infty}\frac{1}{[t]}\int_X a_{[t]}\,d\mu=\int_X\tilde a\,d\mu=\int_X\lim_{t\to\infty}\frac{a_{[t]}}{[t]}\,d\mu. \tag{7}$$

Finally, by (5) we have

$$\begin{aligned}\left|\frac1t\int_X a_t\,d\mu-\frac{[t]}{t}\frac{1}{[t]}\int_X a_{[t]}\,d\mu\right|&\le\left|\frac1t\int_X a_{t-[t]}\circ\phi_{[t]}\,d\mu\right|+\frac{C}{t}\mu(X)\\&\le\frac1t\mu(X)\sup_{s\in[0,1]}\|a_s\|_\infty+\frac{C}{t}\mu(X)\end{aligned}$$

and so, using (7), we obtain

$$\lim_{t\to\infty}\frac1t\int_X a_t\,d\mu=\lim_{t\to\infty}\frac{[t]}{t}\frac{1}{[t]}\int_X a_{[t]}\,d\mu=\int_X\lim_{t\to\infty}\frac{a_t(x)}{t}\,d\mu(x).$$

This shows that the two limits in (4) exist and are equal.

Now we establish the inequality

$$P(a)\le\sup_{\mu\in\mathcal{M}}\left(h_\mu(\Phi)+\lim_{t\to\infty}\frac1t\int_X a_t\,d\mu\right). \tag{8}$$

First we obtain a few auxiliary results. Given $x\in X$, we define a probability measure on X by

$$\mu_{x,t}=\frac1t\int_0^t\delta_{\phi_s(x)}\,ds,$$

where δ_y is the probability measure concentrate on y. Let also $V(x)$ be the set of all sublimits of the family $(\mu_{x,t})_{t>0}$.

Lemma 4. *Given $x\in X$ and $\mu\in V(x)$, there exists an increasing sequence $(t_n)_{n\in\mathbb{N}}$ such that*

$$\lim_{n\to\infty}\frac{a_{t_n}(x)}{t_n}=\lim_{t\to\infty}\frac1t\int_X a_t\,d\mu.$$

Proof of the lemma. Let $(t_n)_{n\in\mathbb{N}}$ be an increasing sequence such that the sequence of measures $(\mu_{x,t_n})_{n\in\mathbb{N}}$ converges to μ. Given $\varepsilon>0$, there exist $K>0$ and $C_{\varepsilon,k}>0$ for each $k\ge K$ such that

$$\left|a_n(x)-\frac1k\sum_{j=0}^{n-1}a_k(\phi_j(x))\right|\le n\varepsilon+C_{\varepsilon,k}$$

for every $n>2k$ (see [9]). This implies that

$$\left|\frac{a_n(x)}{n}-\frac1k\int_X a_k\,d\mu_{x,n}\right|=\left|\frac{a_n(x)}{n}-\frac{1}{kn}\sum_{j=0}^{n-1}a_k(\phi_j(x))\right|\le\varepsilon+\frac{C_{\varepsilon,k}}{n}. \tag{9}$$

Using the sequence t_n instead of n in (9) and letting $n \to \infty$, we obtain

$$\frac{1}{k}\int_X a_k\,d\mu - \varepsilon \le \varliminf_{n\to\infty} \frac{a_{t_n}(x)}{t_n} \le \varlimsup_{n\to\infty} \frac{a_{t_n}(x)}{t_n} \le \frac{1}{k}\int_X a_k\,d\mu + \varepsilon. \tag{10}$$

Finally, letting $k \to \infty$ in (10) gives

$$\lim_{k\to\infty}\frac{1}{k}\int_X a_k\,d\mu - \varepsilon \le \varliminf_{n\to\infty} \frac{a_{t_n}(x)}{t_n} \le \varlimsup_{n\to\infty} \frac{a_{t_n}(x)}{t_n} \le \lim_{k\to\infty}\frac{1}{k}\int_X a_k\,d\mu + \varepsilon$$

and it follows from the arbitrariness of ε that

$$\lim_{n\to\infty} \frac{a_{t_n}(x)}{t_n} = \lim_{k\to\infty}\frac{1}{k}\int_X a_k\,d\mu = \lim_{t\to\infty}\frac{1}{t}\int_X a_t\,d\mu.$$

This completes the proof of the lemma. □

We also need the following technical property (see [6] for a corresponding result in the additive case).

Lemma 5. *Let $\Gamma \subset X \times \{1\}$ be a finite cover of X. For the open cover $\mathcal{V} = \{V_1, \dots, V_r\}$ of X, where $V_j = B_1(x_j, \varepsilon/2)$ with $(x_j, 1) \in \Gamma$, there exist $m, T \in \mathbb{N}$ with T arbitrary large and a sequence $U = V_{i_1} \cdots V_{i_T}$ such that:*

(1) *$x \in \bigcap_{r=1}^{T} \phi_{-r+1} V_{i_r}$ and*

$$a_T(x) \le T\left(\lim_{t\to\infty}\frac{1}{t}\int_X a_t\,d\mu + \delta\right);$$

(2) *there exists a subset $V \in (\mathcal{V}^m)^k$ of U of length $km \ge T - m$ such that $H(V) \le m(h_\mu(\Phi) + \delta)$.*

Proof of the lemma. By Lemma 4, given $\delta > 0$, we have

$$\left|\frac{a_{t_n}(x)}{t_n} - \lim_{t\to\infty}\frac{1}{t}\int_X a_t\,d\mu\right| < \delta$$

for any sufficiently large n. So one can take T arbitrarily large such that

$$a_T(x) \le T\left(\lim_{t\to\infty}\frac{1}{t}\int_X a_t\,d\mu + \delta\right)$$

and the first property follows. The second property can be obtained as in the proof of Lemma 4.3.2 in [4]. □

Given $\delta > 0$, $m \in \mathbb{N}$ and $u \in \mathbb{R}$, let $X_{m,u}$ be the set of points $x \in X$ satisfying the two properties in Lemma 5 for some measure $\mu \in V(x)$ with

$$u - \delta \le \lim_{t\to\infty}\frac{1}{t}\int_X a_t\,d\mu \le u + \delta.$$

Moreover, let n_T be the number of all sequences $U \in \mathcal{V}^T$ with these two properties for some point $x \in X_{m,u}$. Proceeding as in the proof Lemma 4.3.3 in [4], we find that

$$n_T \le \exp[T\,(h_\mu(\Phi) + 2\delta)]$$

for any sufficiently large T. Therefore,

$$
\begin{aligned}
&M(X_{m,u}, a, \alpha, \varepsilon)\\
&\le \varlimsup_{\tau\to+\infty} \sum_{T=\tau}^{+\infty} n_T \exp\left[-\alpha T + T\left(\lim_{t\to\infty}\frac{1}{t}\int_X a_t\,d\mu + \delta\right) + \gamma_T(a,\varepsilon)\right]\\
&\le \varlimsup_{\tau\to+\infty} \sum_{T=\tau}^{+\infty} \exp\left[T\left(h_\mu(\Phi) + \lim_{t\to\infty}\frac{1}{t}\int_X a_t\,d\mu + 3\delta - \alpha + \varlimsup_{t\to+\infty}\frac{\gamma_t(a,\varepsilon)}{t}\right)\right]\\
&\le \varlimsup_{\tau\to+\infty} \sum_{T=\tau}^{+\infty} \beta^T,
\end{aligned}
\tag{11}
$$

where

$$\beta = \exp\left(-\alpha + c + 3\delta + \varlimsup_{t\to+\infty}\frac{\gamma_t(a,\varepsilon)}{t}\right)$$

and

$$c = \sup_{\mu\in\mathcal{M}}\left(h_\mu(\Phi) + \lim_{t\to\infty}\frac{1}{t}\int_X a_t\,d\mu\right).$$

For

$$\alpha > c + 3\delta + \varlimsup_{t\to+\infty}\frac{\gamma_t(a,\varepsilon)}{t} \tag{12}$$

we have $\beta < 1$ and so it follows from (11) that

$$M(X_{m,u}, a, \alpha, \varepsilon) \le \varlimsup_{\tau\to+\infty}\sum_{T=\tau}^{+\infty}\beta^T = 0 \quad\text{and}\quad \alpha > P(a|_{X_{m,u}}, \varepsilon). \tag{13}$$

Recall that $\tilde{a} = \lim_{t\to\infty}(a_t/t)$. Taking points $u_1, \ldots, u_r$ such that for each $u \in [\inf\tilde{a}, \sup\tilde{a}]$ there exists $j \in \{1, \ldots, r\}$ with $|u - u_j| < \delta$, we have

$$X = \bigcup_{m\in\mathbb{N}}\bigcup_{i=1}^{r} X_{m,u_i}.$$

Finally, by (12) and (13), we obtain

$$
\begin{aligned}
P(a) &= \varlimsup_{\varepsilon\to 0} P(a,\varepsilon)\\
&= \varlimsup_{\varepsilon\to 0}\sup_{m,i} P(a|_{X_{m,u_i}}, \varepsilon)\\
&\le c + \varlimsup_{\varepsilon\to 0}\varlimsup_{t\to\infty}\frac{\gamma_t(a,\varepsilon)}{t} + 3\delta = c + 3\delta.
\end{aligned}
$$

Since δ is arbitrary, we conclude that $P(a) \le c$ and so inequality (8) holds.

To obtain the reverse inequality $P(a) \ge c$, to the possible extent we follow corresponding arguments in [4].

Lemma 6. *For each ergodic measure $\mu \in \mathcal{M}$, we have*

$$P(a) \ge h_\mu(\Phi) + \lim_{t\to\infty}\frac{1}{t}\int_X a_t\,d\mu.$$

Proof of the lemma. Given $\varepsilon > 0$, there exist $\delta \in (0, \varepsilon)$, a measurable partition $\xi = \{C_1, \ldots, C_m\}$ of X and an open cover $\mathcal{V} = \{V_1, \ldots, V_k\}$ of X for some $k \ge m$ such that:

(1) $\operatorname{diam} C_j \le \varepsilon$, $\overline{V_i} \subset C_i$ and $\mu(C_i \setminus V_i) < \delta^2$ for $i = 1, \dots, m$;
(2) the set $E = \bigcup_{i=m+1}^k V_i$ has measure $\mu(E) < \delta^2$.

We consider a measure ν in the ergodic decomposition of μ with respect to the time-1 map ϕ_1. The ergodic decomposition is described by a measure τ in the space $\mathcal{M}'$ of ϕ_1-invariant probability measures that is concentrated on the ergodic measures (with respect to ϕ_1). Note that $\nu(E) < \delta$ for ν in a set $\mathcal{M}_\delta \subset \mathcal{M}'$ of positive τ-measure such that $\tau(\mathcal{M}_\delta) \to 1$ when $\delta \to 0$.

For each $x \in X$ and $n \in \mathbb{N}$, let $s_n(x)$ be the number of integers $l \in [0, n)$ such that $\phi_1^l(x) \in E$. By Birkhoff's ergodic theorem, since ν is ergodic for ϕ_1 we have

$$\lim_{n\to+\infty} \frac{s_n(x)}{n} = \lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \chi_E(\phi_1^j(x)) = \int_X \chi_E \, d\nu = \nu(E) \tag{14}$$

for ν-almost every $x \in X$. On the other hand, by Lemma 4, there exists an increasing sequence of integers $(t_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} \frac{a_{t_n}(x)}{t_n} = \lim_{t\to\infty} \frac{1}{t} \int_X a_t \, d\mu \tag{15}$$

for μ-almost every $x \in X$. By (14) and (15) together with Egorov's theorem, there exist $\nu \in \mathcal{M}_\delta$, $N_1 \in \mathbb{N}$ and a measurable set $A_1 \subset X$ with $\nu(A_1) \ge 1-\delta$ such that

$$\frac{s_n(x)}{n} < 2\delta \quad \text{and} \quad \left| \frac{a_n(x)}{n} - \lim_{t\to\infty} \frac{1}{t} \int_X a_t \, d\mu \right| < \delta \tag{16}$$

for every $x \in A_1$ and $n > N_1$. For the partition

$$\xi_n := \bigvee_{j=0}^{n} \phi_1^{-j}(\xi),$$

one can use the Shannon–McMillan–Breiman theorem and again Egorov's theorem to conclude that there exist $N_2 \in \mathbb{N}$ and a measurable set $A_2 \subset X$ with $\nu(A_2) \ge 1 - \delta$ such that

$$\nu(\xi_n(x)) \le \exp\left[(-h_\nu(\phi_1, \xi) + \delta)n\right] \tag{17}$$

for every $x \in A_2$ and $n > N_2$, where $\xi_n(x)$ is the element of ξ_n containing x. We take $N = \max\{N_1, N_2\}$ and $A = A_1 \cap A_2$. Then $\nu(A) \ge 1 - 2\delta$ and by construction, (16) and (17) hold for every $x \in A$ and $n > N$.

Let Δ be a Lebesgue number of the cover $\mathcal{V}$ and take $\overline{\varepsilon} > 0$ with $2\overline{\varepsilon} < \Delta$. Given $\alpha \in \mathbb{R}$, take $\overline{N} \ge N$ such that for each $n \ge \overline{N}$ there exists a set $\Gamma \subset X \times [n, +\infty)$ covering X with

$$\left| \sum_{(x,t)\in\Gamma} \exp(a(x,t,\overline{\varepsilon}) - \alpha t) - M(X, a, \alpha, \overline{\varepsilon}) \right| < \delta. \tag{18}$$

Without loss of generality, we also assume that $\overline{N}$ is so large such that

$$\frac{\gamma_l(a, \overline{\varepsilon})}{l} \le \overline{\lim_{t\to+\infty}} \frac{\gamma_t(a, \overline{\varepsilon})}{t} + \delta$$

for all $l \ge \overline{N}$. Moreover, given $l \in \mathbb{N}$, let

$$\Gamma_l = \left\{(x,t) \in \Gamma : B_l(x, \overline{\varepsilon}) \cap A \ne \emptyset\right\}$$

and let $B_l = \bigcup_{(x,t)\in\Gamma} B_t(x,\overline{\varepsilon})$. Following arguments in [4], it follows from the first inequality in (16) and (17) that

$$\operatorname{card}\Gamma_l \geq \nu(B_l \cap A)\exp\bigl[h_\nu(\phi_1,\xi)l - (1+2\log\operatorname{card}\xi)l\delta\bigr]$$

for $l \in \mathbb{N}$. On the other hand, by the second inequality in (16) we have

$$\sup_{B_l(x,\overline{\varepsilon})} a_l \geq l\left(\lim_{t\to\infty}\frac{1}{t}\int_X a_t\,d\mu - \delta\right) - \gamma_l(a,\overline{\varepsilon})$$

for all $l \geq \overline{N}$ and $(x,t) \in \Gamma_l$. Then

$$\begin{aligned}
&\sum_{(x,t)\in\Gamma} \exp(a(x,t,\overline{\varepsilon}) - \alpha t)\\
&\geq \sum_{l=\overline{N}}^{+\infty}\sum_{(x,t)\in\Gamma_l}\exp\Bigl(\sup_{B_l(x,\overline{\varepsilon})} a_l - \alpha l\Bigr)\\
&\geq \sum_{l=\overline{N}}^{+\infty}\operatorname{card}\Gamma_l\exp\left[\left(-\alpha + \lim_{t\to\infty}\frac{1}{t}\int_X a_t\,d\mu - \delta\right)l - \gamma_l(a,\overline{\varepsilon})\right]\\
&\geq \sum_{l=\overline{N}}^{+\infty}\nu(B_l\cap A)\\
&\times\exp\left[\left(h_\nu(\phi_1,\xi) + \lim_{t\to\infty}\frac{1}{t}\int_X a_t\,d\mu - \frac{\gamma_l(a,\overline{\varepsilon})}{l} - \alpha\right)l - 2(1+\log\operatorname{card}\xi)l\delta\right].
\end{aligned}$$

Taking

$$\alpha < h_\nu(\phi_1,\xi) + \lim_{t\to\infty}\frac{1}{t}\int_X a_t\,d\mu - \varlimsup_{t\to+\infty}\frac{\gamma_t(a,\overline{\varepsilon})}{t}$$

and assuming that δ is so small such that

$$\alpha < h_\nu(\phi_1,\xi) + \lim_{t\to\infty}\frac{1}{t}\int_X a_t\,d\mu - \varlimsup_{t\to+\infty}\frac{\gamma_t(a,\overline{\varepsilon})}{t} - 2(1+\log\operatorname{card}\xi)\delta - \delta,$$

we finally obtain

$$\sum_{(x,t)\in\Gamma}\exp(a(x,t,\overline{\varepsilon}) - \alpha t) \geq \sum_{l=\overline{N}}^{+\infty}\nu(B_l\cap A) \geq \nu(A) \geq 1-2\delta.$$

Hence, it follows from (18) that

$$M(X,a,\alpha,\overline{\varepsilon}) > 1-3\delta > 0$$

and so $P(a,\overline{\varepsilon}) \geq \alpha$, which implies that

$$P(a,\overline{\varepsilon}) \geq h_\nu(\phi_1,\xi) + \lim_{t\to\infty}\frac{1}{t}\int_X a_t\,d\mu - \varlimsup_{t\to+\infty}\frac{\gamma_t(a,\overline{\varepsilon})}{t}.$$

Now we consider measurable partitions ξ_l and open covers $\mathcal{V}_l$ as before with $\varepsilon = 1/l$. For each l, take $\overline{\varepsilon}_l > 0$ such that $2\overline{\varepsilon}_l < 1/l$ is a Lebesgue number of the cover $\mathcal{V}_l$. Since $\operatorname{diam}\xi_l \to 0$ when $l \to +\infty$, we have

$$\lim_{l\to+\infty} h_\nu(\phi_1,\xi_l) = h_\nu(\phi_1).$$

Since the family a has tempered variation, we obtain

$$\begin{aligned}P(a) &= \lim_{l\to+\infty} P(a,\overline{\varepsilon}_l)\\ &\ge \lim_{l\to+\infty} h_\nu(\phi_1,\xi_l) + \lim_{t\to\infty}\frac{1}{t}\int_X a_t\,d\mu - \lim_{l\to+\infty}\overline{\lim_{t\to+\infty}}\frac{\gamma_t(a,\overline{\varepsilon}_l)}{t}\\ &= h_\nu(\phi_1) + \lim_{t\to\infty}\frac{1}{t}\int_X a_t\,d\mu.\end{aligned}$$

Integrating with respect to ν gives

$$P(a) \ge \int_{\mathcal{M}_\delta} h_\nu(\phi_1)\,d\tau(\nu) + \lim_{t\to\infty}\frac{1}{t}\int_X a_t\,d\mu$$

and letting $\delta \to 0$ yields the inequality

$$\begin{aligned}P(a) &\ge \int_{\mathcal{M}'} h_\nu(\phi_1)\,d\tau(\nu) + \lim_{t\to\infty}\frac{1}{t}\int_X a_t\,d\mu\\ &= h_\mu(\phi_1) + \int_Z b\,d\mu = h_\mu(\Phi) + \lim_{t\to\infty}\frac{1}{t}\int_X a_t\,d\mu.\end{aligned}$$

This completes the proof of the lemma. □

Now we consider the set

$$X_\mu = \big\{x \in X : V(x) = \{\mu\}\big\}.$$

When $\mu \in \mathcal{M}$ is ergodic, X_μ is a nonempty Φ-invariant set and $\mu(X_\mu) = 1$. Hence, it follows from Lemma 6 that

$$P(a) \ge P(a|_{X_\mu}) \ge h_\mu(\Phi|_{X_\mu}) + \lim_{t\to\infty}\frac{1}{t}\int_{X_\mu} a_t\,d\mu = h_\mu(\Phi) + \lim_{t\to\infty}\frac{1}{t}\int_X a_t\,d\mu.$$

When $\mu \in \mathcal{M}$ is arbitrary, we can decompose X into ergodic components and use the previous argument to show that

$$P(a) \ge \sup_{\mu\in\mathcal{M}}\left(h_\mu(\Phi) + \lim_{t\to\infty}\frac{1}{t}\int_X a_t\,d\mu\right).$$

This completes the proof of the theorem. □

We say that a Φ-invariant measure μ_a is an *equilibrium measure* for the almost additive family a (with respect to the flow Φ) if the supremum in (4) is attained at μ_a, that is, if

$$P(a) = h_{\mu_a}(\Phi) + \lim_{t\to\infty}\frac{1}{t}\int_X a_t\,d\mu_a. \tag{19}$$

The following result gives a criterion for the existence of equilibrium measures in this context.

Theorem 7. *Let Φ be a continuous flow on a compact metric space X such that the map $\mu \mapsto h_\mu(\Phi)$ is upper semicontinuous. Then each almost additive family a of continuous functions with tempered variation has at least one equilibrium measure.*

Proof. Since a_n+C is a subadditive sequence of functions, the real sequence $\int_X(a_n+C)\,d\mu$ is also subadditive. Then

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\int_X a_n\,d\mu &= \lim_{n\to\infty}\frac{1}{n}\int_X (a_n+C)\,d\mu \\ &\le \frac{1}{n}\int_X (a_n+C)\,d\mu = \frac{1}{n}\int_X a_n\,d\mu + \frac{C}{n}.\end{aligned} \tag{20}$$

Similarly, the sequence $\int_X(a_n-C)\,d\mu$ is supadditive and so

$$\lim_{n\to\infty}\frac{1}{n}\int_X a_n\,d\mu \ge \frac{1}{n}\int_X a_n\,d\mu - \frac{C}{n}. \tag{21}$$

It follows from (20) and (21) that

$$\left|\lim_{n\to\infty}\frac{1}{n}\int_X a_n\,d\mu - \frac{1}{n}\int_X a_n\,d\mu\right| \le \frac{C}{n}. \tag{22}$$

Now let μ_m be a sequence of measures converging to μ. Then

$$\left|\lim_{n\to\infty}\frac{1}{n}\int_X a_n\,d\mu_m - \frac{1}{n}\int_X a_n\,d\mu_m\right| \le \frac{C}{n}$$

for every $m,n\in\mathbb{N}$. Letting $m\to\infty$ and then $n\to\infty$, we obtain

$$\lim_{m\to\infty}\lim_{n\to\infty}\frac{1}{n}\int_X a_n\,d\mu_m = \lim_{n\to\infty}\frac{1}{n}\int_X a_n\,d\mu.$$

This shows that the map

$$\mu\mapsto\lim_{t\to\infty}\frac{1}{t}\int_X a_t\,d\mu$$

is continuous for each almost additive family a. Together with the upper semicontinuity of the map $\mu\mapsto h_\mu(\Phi)$, this implies that the map

$$\mu\mapsto h_\mu(\Phi)+\lim_{t\to\infty}\frac{1}{t}\int_X a_t\,d\mu$$

is upper semicontinuous. Hence, in view of the compactness of $\mathcal{M}$ there exists a measure $\mu_a\in\mathcal{M}$ satisfying (19). □

3. Hyperbolic flows

In this section we consider the particular case of hyperbolic flows and we describe a general condition for the uniqueness of the equilibrium measure of an almost additive family of continuous functions with tempered variation as well as for its Gibbs property.

3.1. **Basic notions.** Let $\Phi=(\phi_t)_{t\in\mathbb{R}}$ be a C^1 flow on a smooth manifold M. A compact Φ-invariant set $\Lambda\subset M$ is called a *hyperbolic set for* Φ if there exists a splitting

$$T_\Lambda M = E^s\oplus E^u\oplus E^0$$

and constants $c>0$ and $\lambda\in(0,1)$ such that for each $x\in\Lambda$:

(1) the vector $(d/dt)\phi_t(x)|_{t=0}$ generates $E^0(x)$;
(2) for each $t\in\mathbb{R}$ we have

$$d_x\phi_t E^s(x)=E^s(\phi_t(x)) \quad\text{and}\quad d_x\phi_t E^u(x)=E^u(\phi_t(x));$$

(3) $\|d_x\phi_t v\|\le c\lambda^t\|v\|$ for $v\in E^s(x)$ and $t>0$;

(4) $\|d_x\phi_{-t}v\| \le c\lambda^t\|v\|$ for $v \in E^u(x)$ and $t > 0$.

Given a hyperbolic set Λ for a flow Φ, for each $x \in \Lambda$ and any sufficiently small $\varepsilon > 0$ we define

$$A^s(x) = \big\{y \in B(x,\varepsilon) : d(\phi_t(y), \phi_t(x)) \searrow 0 \text{ when } t \to +\infty\big\}$$

and

$$A^u(x) = \big\{y \in B(x,\varepsilon) : d(\phi_t(y), \phi_t(x)) \searrow 0 \text{ when } t \to -\infty\big\}.$$

Moreover, let $V^s(x) \subset A^s(x)$ and $V^u(x) \subset A^u(x)$ be the largest connected components containing x. These are smooth manifolds, called respectively *(local) stable and unstable manifolds* of size ε at the point x, satisfying:

(1) $T_xV^s(x) = E^s(x)$ and $T_xV^u(x) = E^u(x)$;

(2) for each $t > 0$ we have

$$\phi_t(V^s(x)) \subset V^s(\phi_t(x)) \quad \text{and} \quad \phi_{-t}(V^u(x)) \subset V^u(\phi_{-t}(x));$$

(3) there exist $\kappa > 0$ and $\mu \in (0,1)$ such that for each $t > 0$ we have

$$d(\phi_t(y), \phi_t(x)) \le \kappa\mu^t d(y,x) \quad \text{for } y \in V^s(x)$$

and

$$d(\phi_{-t}(y), \phi_{-t}(x)) \le \kappa\mu^t d(y,x) \quad \text{for } y \in V^u(x).$$

We recall that a set Λ is said to be *locally maximal* (with respect to a flow Φ) if there exists an open neighborhood U of Λ such that

$$\Lambda = \bigcap_{t\in\mathbb{R}} \phi_t(U).$$

Given a locally maximal hyperbolic set Λ and a sufficiently small $\varepsilon > 0$, there exists $\delta > 0$ such that if $x, y \in \Lambda$ satisfy $d(x,y) \le \delta$, then there exists a unique $t = t(x,y) \in [-\varepsilon, \varepsilon]$ such that

$$[x,y] := V^s(\phi_t(x)) \cap V^u(x)$$

is a single point in Λ.

Now we make some preparations to introduce the notion of a Markov system. Consider an open smooth disk $D \subset M$ of dimension $\dim M - 1$ that is transverse to Φ and take $x \in D$. Let $U(x)$ be an open neighborhood of x diffeomorphic to $D \times (-\varepsilon, \varepsilon)$. Then the projection $\pi_D \colon U(x) \to D$ defined by $\pi_D(\phi_t(y)) = y$ is differentiable. We say that a closed set $R \subset \Lambda \cap D$ is a *rectangle* if $R = \overline{\operatorname{int} R}$ and $\pi_D([x,y]) \in R$ for $x, y \in R$.

Consider rectangles $R_1, \ldots, R_k \subset \Lambda$ (each contained in some open smooth disk transverse to the flow) such that

$$R_i \cap R_j = \partial R_i \cap \partial R_j \quad \text{for } i \ne j.$$

Let $Z = \bigcup_{i=1}^k R_i$. We assume that there exists $\varepsilon > 0$ such that:

(1) $\Lambda = \bigcup_{t\in[0,\varepsilon]} \phi_t(Z)$;

(2) whenever $i \ne j$, either

$$\phi_t(R_i) \cap R_j = \emptyset \quad \text{for all } t \in [0,\varepsilon]$$

or

$$\phi_t(R_j) \cap R_i = \emptyset \quad \text{for all } t \in [0,\varepsilon].$$

We define the *transfer function* $\tau\colon \Lambda \to \mathbb{R}_0^+$ by

$$\tau(x) = \min\{t > 0 : \phi_t(x) \in Z\},$$

and the *transfer map* $T\colon \Lambda \to Z$ by

$$T(x) = \phi_{\tau(x)}(x). \tag{23}$$

The restriction T_Z of T to Z is invertible and we have $T^n(x) = \phi_{\tau_n(x)}(x)$, where

$$\tau_n(x) = \sum_{i=0}^{n-1} \tau(T^i(x)).$$

The collection $R_1, \dots, R_k$ is said to be a *Markov system* for Φ on Λ if

$$T(\operatorname{int}(V^s(x) \cap R_i)) \subset \operatorname{int}(V^s(T(x)) \cap R_j)$$

and

$$T^{-1}(\operatorname{int}(V^u(T(x)) \cap R_j)) \subset \operatorname{int}(V^u(x) \cap R_i)$$

for every $x \in \operatorname{int} T(R_i) \cap \operatorname{int} R_j$ and $i, j = 1, \dots, k$. By work of Bowen [5] and Ratner [14], any locally maximal hyperbolic set Λ has Markov systems of arbitrarily small diameter and the transfer function τ is Hölder continuous on each domain of continuity.

Given a Markov system $R_1, \dots, R_k$ for a flow Φ on a locally maximal hyperbolic set Λ, we consider the $k \times k$ matrix A with entries

$$a_{ij} = \begin{cases} 1 & \text{if } \operatorname{int} T(R_i) \cap R_j \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

where T is the map in (23). We also consider the set

$$\Sigma_A = \bigl\{(\cdots i_{-1} i_0 i_1 \cdots) : a_{i_n i_{n+1}} = 1 \text{ for } n \in \mathbb{Z}\bigr\} \subset \{1, \dots, k\}^{\mathbb{Z}}$$

and the shift map $\sigma\colon \Sigma_A \to \Sigma_A$ defined by $\sigma(\cdots i_0 \cdots) = (\cdots j_0 \cdots)$, where $j_n = i_{n+1}$ for each $n \in \mathbb{Z}$. We denote by Σ_n the set of Σ_A-admissible sequences of length n, that is, the finite sequences $(i_1 \cdots i_n)$ for which there exists $(\cdots j_0 j_1 j_2 \cdots) \in \Sigma_A$ such that $(i_1 \dots i_n) = (j_1 \cdots j_n)$. Finally, we define a *coding map* $\pi\colon \Sigma_A \to Z$ by

$$\pi(\cdots i_0 \cdots) = \bigcap_{n \in \mathbb{Z}} R_{i_{-n} \cdots i_n},$$

where $R_{i_{-n} \cdots i_n} = \bigcap_{l=-n}^n \overline{T_Z^{-l} \operatorname{int} R_{i_l}}$. The following properties hold:

1. $\pi \circ \sigma = T \circ \pi$;
2. π is Hölder continuous and onto;
3. π is one-to-one on a full measure set with respect to any ergodic measure of full support and on a residual set.

Given $\beta > 1$, we equip Σ_A with the distance d_β defined by

$$d_\beta(\omega, \omega') = \begin{cases} \beta^{-n} & \text{if } \omega \neq \omega', \\ 0 & \text{if } \omega = \omega', \end{cases}$$

where $n = n(\omega, \omega') \in \mathbb{N} \cup \{0\}$ is the smallest integer such that $i_n(\omega) \neq i_n(\omega')$ or $i_{-n}(\omega) \neq i_{-n}(\omega')$. One can always choose β so that $\tau \circ \pi$ is Lipschitz.

Now let ν be a T_Z-invariant probability measure on Z. One can show that ν induces a Φ-invariant probability measure μ on Λ such that

$$\int_\Lambda g\,d\mu = \frac{\int_Z \int_0^{\tau(x)} (g\circ\phi_s)(x)\,ds\,d\nu}{\int_Z \tau\,d\nu} \tag{24}$$

for any continuous function $g\colon \Lambda\to\mathbb{R}$. In fact, any Φ-invariant probability measure μ on Λ is of this form for some T_Z-invariant probability measure ν on Z. Abramov's entropy formula says that

$$h_\mu(\Phi) = \frac{h_\nu(T_Z)}{\int_Z \tau\,d\nu}. \tag{25}$$

By (24) and (25) we obtain

$$h_\mu(\Phi) + \int_\Lambda g\,d\mu = \frac{h_\nu(T_Z) + \int_Z I_g\,d\nu}{\int_Z \tau\,d\nu}, \tag{26}$$

where $I_g(x) = \int_0^{\tau(x)} (g\circ\phi_s)ds$. Since $\tau > 0$ on Z, it follows from (26) that

$$P_\Phi(g) = 0 \quad \text{if and only if} \quad P_{T_Z}(I_g) = 0,$$

where $P_\Phi(g)$ is the topological pressure of g with respect to Φ and $P_{T_Z}(I_g)$ is the topological pressure of I_g with respect to the map T_Z. When $P_\Phi(g) = 0$, this implies that μ is an equilibrium measure for g if and only if ν is an equilibrium measure for I_g.

3.2. **Technical preparations.** In this section we make a few technical preparations. We start by considering the sequence of functions $a_n\colon Z\to\mathbb{R}$ defined by

$$c_n(x) = a_{\tau_n(x)}(x). \tag{27}$$

Lemma 8. *The sequence $c = (c_n)_{n\in\mathbb{N}}$ is almost additive with respect to the map T_Z.*

Proof. Notice that

$$c_{n+m}(x) = a_{\tau_{n+m}(x)}(x) = a_{\tau_n(x)+\tau_m(T^n(x))}(x) \tag{28}$$

for $n, m \in \mathbb{N}$. Since a is almost additive with respect to Φ, by (28) we have

$$\begin{aligned} c_{n+m}(x) &\le a_{\tau_n(x)}(x) + a_{\tau_m(T^n(x))}(\phi_{\tau_n(x)}(x)) + C \\ &= a_{\tau_n(x)}(x) + a_{\tau_m(T^n(x))}(T^n(x)) + C \\ &= c_n(x) + c_m(T^n(x)) + C. \end{aligned}$$

Similarly, we have also

$$\begin{aligned} c_{n+m}(x) &\ge a_{\tau_n(x)}(x) + a_{\tau_m(T^n(x))}(T^n(x)) - C \\ &= c_n(x) + c_m(T^n(x)) - C. \end{aligned}$$

This shows that c is an almost additive sequence with respect to T_Z. □

Now we consider the sets $B_t(x,\varepsilon)$ in (2) with $X = \Lambda$.

Lemma 9. *Given $\delta > 0$, there exists a Markov system $R_1, \dots, R_k$ and a constant $C > 0$ such that*

$$R_{i_{-n}\cdots i_n} \subset B_{\tau_n(y)}(y, C\delta)$$

for every $n \in \mathbb{N}$ and $y \in R_{i_{-n}\cdots i_n}$.

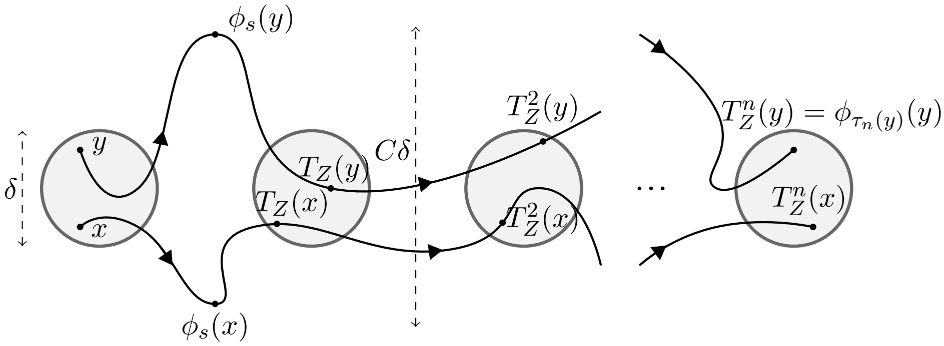

FIGURE 1. $d(\phi_s(x), \phi_s(y)) < C\delta$ for $s \in [0, \tau_n(y)]$.

Proof. Since the rectangles of a Markov system can have arbitrarily small diameters, for each $\delta > 0$ there exist $R_1, \ldots, R_k$ such that

$$R_j \subset B(z, \delta) \quad \text{for every } z \in R_j. \tag{29}$$

Given $x, y \in R_{i_{-n} \cdots i_n}$, we have $T^k(x), T^k(y) \in R_{i_k}$ for $k \in \{0, \ldots, n\}$. On the other hand, by (29), $R_{i_k} \subset B(T^k(y), \delta)$ and so $d(T^k(x), T^k(y)) < \delta$ for $k \in \{0, 1, \ldots, n\}$. Finally, by the uniform continuity of $(t, x) \mapsto \phi_t(x)$ on compact sets, there exists $C = C(\delta) > 0$ (independent of n) such that

$$d(\phi_s(x), \phi_s(y)) < C\delta \quad \text{for } s \in [0, \tau_n(y)]$$

(see Figure 1). This yields the desired statement. □

Given $\delta > 0$ and a Markov system as in Lemma 9, we consider the numbers

$$V_n(c) = \sup\bigl\{|c_n(x) - c_n(y)| : x, y \in R_{i_1 \cdots i_n}\bigr\}$$

for $n \in \mathbb{N}$. We recall that a family of functions a is said to have *bounded variation* if for every $\kappa > 0$ there exists $\varepsilon > 0$ such that

$$|a_t(x) - a_t(y)| < \kappa \quad \text{whenever } y \in B_t(x, \varepsilon).$$

We shall always assume that $C\delta < \varepsilon$.

Lemma 10. *If a has bounded variation and $\sup_{t \in [0,s]} \|a_t\|_\infty < \infty$ for some $s > 0$, then $\sup_{n \in \mathbb{N}} V_n(c) < \infty$ (in particular, c has tempered variation).*

Proof. Take $x, y \in R_{i_{-n} \cdots i_n}$ and $\omega, \omega' \in \Sigma_A$ such that $T(x) = \pi(\sigma(\omega))$ and $T(y) = \pi(\sigma(\omega'))$. Choosing $\beta > 1$ so that $\tau \circ \pi$ is Lipschitz, say with Lipschitz constant $L > 0$, one can write

$$\begin{aligned}
|\tau_n(x) - \tau_n(y)| &= \left| \sum_{l=0}^{n-1} \tau(T^l(x)) - \sum_{l=0}^{n-1} \tau(T^l(y)) \right| \\
&\le \sum_{l=0}^{n-1} |(\tau \circ \pi)(\sigma^l(\omega)) - (\tau \circ \pi)(\sigma^l(\omega'))| \\
&\le \sum_{l=0}^{n-1} L d_\beta(\sigma^l(\omega), \sigma^l(\omega')).
\end{aligned}$$

This implies that there exists $D > 0$ (independent of x, y and n) such that

$$|\tau_n(x) - \tau_n(y)| \le D. \tag{30}$$

Assuming without loss of generality that $\tau_n(x) > \tau_n(y)$, since the family a is almost additive, we have

$$a_{\tau_n(x)}(x) \le a_{\tau_n(x)-\tau_n(y)}(x) + a_{\tau_n(y)}(\phi_{\tau_n(x)-\tau_n(y)}(x)) + C.$$

Together with (30), this implies that

$$\begin{aligned} c_n(x) - c_n(y) &= a_{\tau_n(x)}(x) - a_{\tau_n(y)}(y) \\ &\le |a_{\tau_n(x)-\tau_n(y)}(x)| + |a_{\tau_n(y)}(\phi_{\tau_n(x)-\tau_n(y)}(x)) - a_{\tau_n(y)}(y)| + C \\ &\le \sup_{l\in[0,D]} \|a_l\|_\infty + |a_{\tau_n(y)}(\phi_{\tau_n(x)-\tau_n(y)}(x)) - a_{\tau_n(y)}(y)| + C \\ &\le \sup_{l\in[0,D]} \|a_l\|_\infty + |a_{\tau_n(y)}(\phi_{\tau_n(x)-\tau_n(y)}(x)) - a_{\tau_n(y)}(x)| \\ &\quad + |a_{\tau_n(y)}(x) - a_{\tau_n(y)}(y)| + C. \end{aligned} \tag{31}$$

Since a is almost additive and $\sup_{t\in[0,s]} \|a_t\|_\infty < \infty$ for some $s > 0$, we have

$$\sup_{l\in[0,D]} \|a_l\|_\infty \le M \tag{32}$$

for some constant $M > 0$. Moreover, by the definition of bounded variation,

$$|a_{\tau_n(y)}(x) - a_{\tau_n(y)}(y)| \le \kappa. \tag{33}$$

Now note that

$$y = \phi_{-\tau_n(y)}(\phi_{\tau_n(y)}(y)) = \phi_{-\tau_n(y)}(T^n(y))$$

and define

$$z := \phi_{-\tau_n(y)}(\phi_{\tau_n(x)}(x)) = \phi_{-\tau_n(y)}(T^n(x)).$$

Since $T^n(x), T^n(y) \in R_{i_n}$ and the map $p \mapsto \phi_{-\tau_n(y)}(p)$ is uniformly continuous on compact sets, there exists $\delta' > 0$ (depending only on δ) such that $d(y,z) < \delta'$. Thus,

$$d(x,z) \le d(x,y) + d(y,z) < \delta + \delta'.$$

By the uniform continuity of the map $(t,x) \mapsto \phi_t(x)$ on the set $[0,\tau_n(y)] \times \Lambda$, there exists $\varepsilon > 0$ such that $d(\phi_s(x), \phi_s(z)) < \varepsilon$ for $s \in [0,\tau_n(y)]$. Again by the bounded variation property, we have

$$|a_{\tau_n(y)}(z) - a_{\tau_n(y)}(x)| \le \kappa. \tag{34}$$

By (31) together with (32), (33) and (34), we obtain

$$c_n(x) - c_n(y) \le M + 2\kappa + C.$$

Similarly, using the inequality

$$a_{\tau_n(x)}(x) \ge a_{\tau_n(x)-\tau_n(y)}(x) + a_{\tau_n(y)}(\phi_{\tau_n(x)-\tau_n(y)}(x)) - C,$$

one can show that

$$c_n(x) - c_n(y) \ge -(M + 2\kappa + C).$$

This yields the desired statement. □

Now we introduce the notion of a Gibbs measure for a flow in the present context. A measure μ on a hyperbolic set Λ for a flow Φ is called a *Gibbs measure* for a if it is induced by a measure ν on $Z = \bigcup_{i=1}^k R_i$ satisfying

$$K^{-1} \le \frac{\nu(R_{i_{-n}\cdots i_n})}{\exp\left[-2nP_{T_Z}(c) + c_{2n}(x)\right]} \le K$$

for some constant $K \ge 1$, for every $n \in \mathbb{N}$ and $x \in R_{i_{-n}\cdots i_n}$ (recall that $c_n(x) = a_{\tau_n(x)}(x)$ for $n \in \mathbb{N}$ and $x \in Z$). Considering the sets

$$\tilde{R}_{i_1\cdots i_n} = \bigcup_{i_{-n}\cdots i_0} R_{i_{-n}\cdots i_n},$$

one can verify that this is equivalent to require that

$$\tilde{K}^{-1} \le \frac{\nu(\tilde{R}_{i_1\cdots i_n})}{\exp\left[-nP_{T_Z}(c) + c_n(x)\right]} \le \tilde{K} \tag{35}$$

for some constant $\tilde{K} \ge 1$, for every $n \in \mathbb{N}$ and $x \in \tilde{R}_{i_1\cdots i_n}$. If the measure ν is also invariant, then

$$P_{T_Z}(c) - \frac{c_n(x)}{n} - \frac{\log \tilde{K}}{n} \le -\frac{1}{n}\log \nu(\tilde{R}_{i_1\cdots i_n}) \le P_{T_Z}(c) - \frac{c_n(x)}{n} + \frac{\log \tilde{K}}{n},$$

which implies

$$h_\nu(T_Z, x) := \lim_{n\to\infty} -\frac{1}{n}\log \nu(\tilde{R}_{i_1\cdots i_n}) = P_{T_Z}(c) - \lim_{n\to\infty} \frac{c_n(x)}{n}.$$

By the Shannon–McMillan–Breiman theorem, we have

$$\begin{aligned} h_\nu(T_Z) &= \int_Z h_\nu(T_Z, x)\, d\nu(x) \\ &= P_{T_Z}(c) - \int_Z \lim_{n\to\infty} \frac{c_n(x)}{n}\, d\nu(x) \\ &= P_{T_Z}(c) - \lim_{n\to\infty} \frac{1}{n}\int_Z c_n\, d\nu \end{aligned}$$

and so

$$P_{T_Z}(c) = h_\nu(T_Z) + \lim_{n\to\infty} \frac{1}{n}\int_Z c_n\, d\nu.$$

This shows that any invariant Gibbs measure satisfying (35) is an equilibrium measure for c with respect to the map T_Z.

3.3. **Existence of Gibbs measures.** Now we state our main result.

Theorem 11. *Let Λ be a hyperbolic set for a topologically mixing C^1 flow Φ and let a be an almost additive family of continuous functions on Λ with bounded variation such that $P_\Phi(a) = 0$ and $\sup_{t\in[0,s]} \|a_t\|_\infty < \infty$ for some $s > 0$. Then:*

1. *there exists a unique equilibrium measure for a;*
2. *there exists a unique invariant Gibbs measure for a;*
3. *the two measures are equal and are mixing.*

Proof. We always consider a Markov system with sufficiently small diameter as in Lemmas 9 and 10. Let c be the sequence in (27).

Lemma 12. *We have*

$$P_{T_Z}(c) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{i_1\cdots i_n} \exp c_n(x_{i_1\cdots i_n}) \tag{36}$$

for any $x_{i_1\cdots i_n} \in R_{i_1\cdots i_n}$*, for each* $(i_1\cdots i_n) \in \Sigma_n$ *and* $n \in \mathbb{N}$*.*

Proof of the lemma. Since c is almost additive, the family d given by $d_n = c_n + C$ is subadditive. Hence, by Theorem 4.2.6 in [4] (see also [7]) we have

$$P_{T_Z}(d) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{i_1\cdots i_n} \exp \max_{x\in R_{i_1\cdots i_n}} d_n(x). \tag{37}$$

Moreover, since c has tempered variation, the same happens with d. Hence, there exist positive numbers λ_n decreasing to zero when $n\to\infty$ such that

$$d_n(x_{i_1\cdots i_n}) \ge \max_{x\in R_{i_1\cdots i_n}} d_n(x) - n\lambda_n \tag{38}$$

for any $x_{i_1\cdots i_n} \in R_{i_1\cdots i_n}$, $(i_1\cdots i_n) \in \Sigma_n$ and $n \in \mathbb{N}$. By (37) and (38) we obtain

$$\varliminf_{n\to\infty} \frac{1}{n} \log \sum_{i_1\cdots i_n} \exp d_n(x_{i_1\cdots i_n}) \ge P_{T_Z}(d).$$

It also follows directly from (37) that

$$P_{T_Z}(d) \ge \varlimsup_{n\to\infty} \frac{1}{n} \log \sum_{i_1\cdots i_n} \exp d_n(x_{i_1\cdots i_n}).$$

Hence,

$$\begin{aligned} P_{T_Z}(d) &= \lim_{n\to+\infty} \frac{1}{n} \log \sum_{i_1\cdots i_n} \exp d_n(x_{i_1\cdots i_n}) \\ &= \lim_{n\to+\infty} \frac{1}{n} \log \sum_{i_1\cdots i_n} \exp c_n(x_{i_1\cdots i_n}) + C. \end{aligned}$$

Since $P_{T_Z}(d) = P_{T_Z}(c) + C$, we find that (36) holds. $\square$

Now let

$$h_{i_1\cdots i_n} = \max\{\exp c_n(y) : y \in R_{i_1\cdots i_n}\} \quad \text{and} \quad H_n = \sum_{i_1\cdots i_n} h_{i_1\cdots i_n}.$$

Moreover, we define a probability measure ν_n in the algebra generated by the sets $R_{i_1\cdots i_n}$ by

$$\nu_n(R_{i_1\cdots i_n}) = h_{i_1\cdots i_n}/H_n \tag{39}$$

for each $(i_1\cdots i_n) \in \Sigma_n$ and we extend it to the Borel σ-algebra $\mathcal{B}$ of Z. Let $\mathcal{M}_Z(c)$ be the set of all sublimits of the sequence $(\nu_n)_{n\in\mathbb{N}}$. Since Z is compact, $\mathcal{M}_Z(c)$ is sequentially compact and so it is nonempty.

Lemma 13. *Each* $\nu \in \mathcal{M}_Z(c)$ *is a Gibbs measure for* c *with respect to* T_Z*.*

Proof of the lemma. Since Φ is topologically mixing, the same happens to the map T_Z. Hence, there exists $r \in \mathbb{N}$ such that A^r has only positive entries, where A is the transition matrix of the Markov system. On the other hand, by Lemma 8, the sequence c is almost additive and so

$$h_{i_1\cdots i_n j_1\cdots j_{l-n}} \le e^C h_{i_1\cdots i_n} h_{j_1\cdots j_{l-n}}.$$

This implies that

$$\sum_{j_1\cdots j_{l-n}} h_{i_1\cdots i_n j_1\cdots j_{l-n}} \le e^C h_{i_1\cdots i_n} H_{l-n} \tag{40}$$

and so

$$H_l = \sum_{i_1\cdots i_n}\sum_{j_1\cdots j_{l-n}} h_{i_1\cdots i_n j_1\cdots j_{l-n}} \le e^C H_{l-n} \sum_{i_1\cdots i_n} h_{i_1\cdots i_n} = e^C H_{l-n} H_n. \tag{41}$$

Take $k = r-1$. Since A^r has only positive entries, given $(j_1\cdots j_{l-k}) \in \Sigma_{l-k}$, there exists $(p_1\cdots p_k) \in \Sigma_k$ such that $(i_1\cdots i_n p_1\cdots p_k j_1\cdots j_{l-k}) \in \Sigma_{n+l}$. Hence, since c is almost additive, for each $x \in R_{i_1\cdots i_n p_1\cdots p_k j_1\cdots j_{l-k}}$ we have

$$\begin{aligned} c_{n+l}(x) &= c_{n+k+(l-k)}(x) \\ &\ge c_{n+k}(x) + c_{l-k}(T_Z^{n+k}(x)) - C \\ &\ge c_n(x) + c_k(T_Z^n(x)) + c_{l-k}(T_Z^{n+k}(x)) - 2C \end{aligned}$$

and so

$$h_{i_1\cdots i_n p_1\cdots p_k j_1\cdots j_{l-k}} \ge e^{c_n(x)} e^{c_k(T_Z^n(x))} e^{c_{l-k}(T_Z^{n+k}(x))} e^{-2C}. \tag{42}$$

We have also

$$-C - \|c_1\|_\infty \le \frac{c_k(z)}{k} \le C + \|c_1\|_\infty \tag{43}$$

for $z \in Z$ and $k \in \mathbb{N}$. Now assume that x satisfies $e^{c_{l-k}(T_Z^{n+k}(x))} = h_{j_1\cdots j_{l-k}}$. Letting $C_1 = e^{-k(C+\|c_1\|_\infty)}$, it follows from (42) and (43) that

$$\begin{aligned} h_{i_1\cdots i_n m_1\cdots m_k j_1\cdots j_{l-k}} &\ge e^{c_n(x)} C_1 e^{c_{l-k}(T_Z^{n+k}(x))} e^{-2C} \\ &\ge C_1 e^{-2C} h_{i_1\cdots i_n} h_{j_1\cdots j_{l-k}}. \end{aligned}$$

Hence, by (41), we obtain

$$\begin{aligned} \sum_{t_1\cdots t_n} h_{i_1\cdots i_n t_1\cdots t_l} &\ge \sum_{j_1\cdots j_{l-k}} h_{i_1\cdots i_n m_1\cdots m_k j_1\cdots j_{l-k}} \\ &\ge C_1 e^{-2C} h_{i_1\cdots i_n} H_{l-k} \\ &\ge C_1 e^{-3C} h_{i_1\cdots i_n} \frac{H_l}{H_k}. \end{aligned} \tag{44}$$

In particular, taking $C_2 = C_1 e^{-3C}/H_k$, we find that

$$H_{n+l} = \sum_{i_1\cdots i_n}\sum_{t_1\cdots t_n} h_{i_1\cdots i_n t_1\cdots t_l} \ge \sum_{i_1\cdots i_n} C_2 h_{i_1\cdots i_n} H_l = C_2 H_n H_l. \tag{45}$$

Now observe that by (41) the sequence $\log(e^C H_n)$ is subadditive and so

$$\lim_{n\to\infty} \frac{1}{n}\log H_n = \lim_{n\to\infty}\frac{1}{n}\log(e^C H_n) = \inf_{n\in\mathbb{N}} \frac{1}{n}\log(e^C H_n). \tag{46}$$

Similarly, by (45), the sequence $\log(C_2 H_n)$ is supadditive and so

$$\lim_{n\to\infty} \frac{1}{n}\log H_n = \lim_{n\to\infty}\frac{1}{n}\log(C_2 H_n) = \sup_{n\in\mathbb{N}} \frac{1}{n}\log(C_2 H_n). \tag{47}$$

On the other hand, by Lemma 12, we have

$$P_{T_Z}(c) = \lim_{n\to\infty}\frac{1}{n}\log\sum_{i_1\cdots i_n} h_{i_1\cdots i_n} = \lim_{n\to\infty}\frac{1}{n}\log H_n$$

and so it follows from (46) and (47) that

$$P_{T_Z}(c) = \inf_{n\in\mathbb{N}} \frac{1}{n}\log(e^C H_n) = \sup_{n\in\mathbb{N}} \frac{1}{n}\log(C_2 H_n).$$

Therefore,

$$C_2 H_n \le e^{nP_{T_Z}(c)} \le e^C H_n \quad \text{for } n \in \mathbb{N}. \tag{48}$$

Note that ν_l is measure on the algebra generated by the sets $R_{i_1\cdots i_l}$ and it follows from (39) that

$$\nu_l(R_{i_1\cdots i_n}) = \sum_{j_1\cdots j_{l-n}} \frac{h_{i_1\cdots i_n j_1\cdots j_{l-n}}}{H_l}.$$

By (40), (45) and (48) together with Lemma 10 we obtain

$$\begin{aligned}
\nu_l(R_{i_1\cdots i_n}) &\le h_{i_1\cdots i_n}\frac{H_{l-n}}{H_l}e^C \le h_{i_1\cdots i_n}\frac{e^C}{H_n C_2}\\
&\le \frac{e^{2C}}{C_2} h_{i_1\cdots i_n} e^{-nP_{T_Z}(c)}\\
&= \frac{e^{2C}}{C_2}(h_{i_1\cdots i_n} e^{-c_n(x)}) e^{-nP_{T_Z}(c)+c_n(x)}\\
&\le C_3 e^{-nP_{T_Z}(c)+c_n(x)}
\end{aligned} \tag{49}$$

for $x \in R_{i_1\cdots i_n}$, where $C_3 = e^\kappa e^{2C}/C_2$ with κ as in the proof of Lemma 10. Similarly, by (44), (41) and (48) we obtain

$$\begin{aligned}
\nu_l(R_{i_1\cdots i_n}) &\ge h_{i_1\cdots i_n}\frac{H_{l-n}}{H_l}C_2 \ge h_{i_1\cdots i_n}\frac{C_2}{H_n e^C}\\
&\ge \frac{C_2^2}{e^C} h_{i_1\cdots i_n} e^{-nP_{T_Z}(c)}\\
&\ge C_4 e^{-nP_{T_Z}(c)+c_n(x)}
\end{aligned} \tag{50}$$

for $x \in R_{i_1\cdots i_n}$, where $C_4 = C_2^2/e^C$. Finally, we consider a sequence ν_{m_k} converging to ν when $k \to \infty$. Replacing l by m_k in (49) and (50) and letting $k \to \infty$, we obtain

$$C_4 \le \frac{\nu_l(R_{i_1\cdots i_n})}{\exp[-nP_{T_Z}(c) + c_n(x)]} \le C_3.$$

This yields the desired statement. □

Lemma 14. *Any Gibbs measure for c with respect to T_Z is ergodic.*

Proof of the lemma. Let ν be a Gibbs measure for c. For $m > n$ we have

$$R_{i_1\cdots i_n} \cap T_Z^{-m}(R_{j_1\cdots j_l}) = \bigcup_{p_1\cdots p_{m-n}} R_{i_1\cdots i_n p_1\cdots p_{m-n} j_1\cdots j_l}$$

and so

$$\nu(R_{i_1\cdots i_n} \cap T_Z^{-m}(R_{j_1\cdots j_l})) = \sum_{p_1\cdots p_{m-n}} \nu(R_{i_1\cdots i_n p_1\cdots p_{m-n} j_1\cdots j_l}).$$

Proceeding as in the proof of Lemma 13, we obtain

$$\begin{aligned}\nu(R_{i_1\cdots i_n} \cap T_Z^{-m}(R_{j_1\cdots j_l})) &\geq C_5 \sum_{p_1\cdots p_{m-n}} (h_{i_1\cdots i_n p_1\cdots p_{m-n} j_1\cdots j_l}) e^{-(m+l)P_{T_Z}(c)} \\ &\geq C_6 e^{-(m+l)P_{T_Z}(c)} h_{i_1\cdots i_n} h_{j_1\cdots j_l} H_{m-n} \\ &\geq C_7 e^{(m-n)P_{T_Z}(c)} e^{-(m+l)P_{T_Z}(c)} h_{i_1\cdots i_n} h_{j_1\cdots j_l} \\ &= C_7 (h_{i_1\cdots i_n} e^{-nP_{T_Z}(c)})(h_{j_1\cdots j_l} e^{-lP_{T_Z}(c)}) \\ &\geq C_8 \nu(R_{i_1\cdots i_n})\nu(R_{j_1\cdots j_l}),\end{aligned}$$

for some constants $C_5, C_6, C_7, C_8 > 0$. Since the sets $R_{i_1\cdots i_n}$ generate the Borel σ-algebra, each Borel set can be arbitrarily approximated in measure by a disjoint union of sets $R_{p_1\cdots p_m}$ (not necessarily with the same m). Thus,

$$\nu(E \cap T_Z^{-m}(F)) \geq C_8 \nu(E)\nu(F) \tag{51}$$

for any Borel sets $E, F \subset Z$. Now let E be a T_Z-invariant set and take $F = E^c$. Then $\nu(E \cap T_Z^{-m}(F)) = 0$ for every $m \in \mathbb{N}$ and it follows from (51) that either $\nu(E) = 0$ or $\nu(E) = 1$. Hence, the measure ν is ergodic. □

We proceed with the proof of the theorem. By Lemma 13, there exists a Gibbs measure ν for c. Any limit point μ of the sequence

$$\nu_n = \frac{1}{n} \sum_{l=0}^{n-1} \nu \circ T_Z^{-l}$$

is a T_Z-invariant measure. Since

$$\nu(T_Z^{-l}(R_{i_1\cdots i_n})) = \sum_{j_1\cdots j_l} \nu(R_{j_1\cdots j_l i_1\cdots i_n}),$$

one can use analogous arguments to those in Lemma 13 to obtain

$$\tilde{L}\nu(R_{i_1\cdots i_n}) \leq \nu(T_Z^{-l}(R_{i_1\cdots i_n})) \leq L\nu(R_{i_1\cdots i_n})$$

for some constants $L, \tilde{L} > 0$. Therefore,

$$\tilde{L}\nu(R_{i_1\cdots i_n}) \leq \frac{1}{n} \sum_{l=0}^{n-1} \nu(T_Z^{-l}(R_{i_1\cdots i_n})) \leq L\nu(R_{i_1\cdots i_n})$$

and so

$$\tilde{L}\nu(R_{i_1\cdots i_n}) \leq \mu(R_{i_1\cdots i_n}) \leq L\nu(R_{i_1\cdots i_n}) \tag{52}$$

for every $n \in \mathbb{N}$ and $(i_1 \cdots i_n) \in \Sigma_n$. It follows from (52) that μ is a Gibbs measure for c. By Lemma 14, the measure ν is ergodic and so, again by (52), the measure μ is also ergodic. We claim that μ is unique. Indeed, assume that there exists another T_Z-invariant measure $\tilde{\mu}$ satisfying (52). By the Gibbs property, $\tilde{\mu}$ is equivalent to μ and since two equivalent invariant ergodic measures are equal, we conclude that $\mu = \tilde{\mu}$.

Now let ν be an equilibrium measure for c with respect to T_Z and let μ be the unique T_Z-invariant Gibbs measure for c. We claim that $\nu = \mu$. Observe that $\nu = \beta\eta + (1-\beta)\overline{\mu}$ for some constant $\beta \in [0,1]$ and some invariant measures $\eta, \overline{\mu}$ such that $\overline{\mu} \ll \mu$ and $\eta \perp \mu$. By Lemma 14, μ is ergodic and so by Birkhoff's ergodic theorem together with the fact that $\overline{\mu} \ll \mu$ we obtain

$$\int_Z \phi \, d\overline{\mu} = \int_Z \phi \, d\mu$$

for every measurable bounded function $\phi \colon Z \to \mathbb{R}$. In particular, $\overline{\mu} = \mu$. Since η and μ are mutually singular T_Z-invariant probability measures,

$$h_\nu(T_Z) = h_{\beta\eta+(1-\beta)\mu}(T_Z) = \beta h_\eta(T_Z) + (1-\beta) h_\mu(T_Z).$$

The fact that ν is an equilibrium measure for c implies that

$$\begin{aligned}
P_{T_Z}(c) &= h_\nu(T_Z) + \lim_{n\to\infty} \frac{1}{n} \int_Z c_n \, d\nu \\
&= \beta h_\eta(T_Z) + (1-\beta) h_\mu(T_Z) \\
&\quad + \lim_{n\to\infty} \frac{1}{n} \left(\int_Z c_n \beta \, d\eta + \int_Z c_n (1-\beta) \, d\mu \right) \\
&= \beta \left(h_\eta(T_Z) + \lim_{n\to\infty} \frac{1}{n} \int_Z c_n \, d\eta \right) \\
&\quad + (1-\beta) \left(h_\mu(T_Z) + \lim_{n\to\infty} \frac{1}{n} \int_Z c_n \, d\mu \right).
\end{aligned} \tag{53}$$

We already know that any invariant Gibbs measure is an equilibrium measure. Since μ is an equilibrium measure, it follows from (53) that either $\beta = 0$, and so $\nu = \overline{\mu} = \mu$, or η is also an equilibrium measure for c.

We show that η cannot be an equilibrium measure, in view of the assumption $\eta \perp \mu$. Let $E \subset Z$ be a T_Z-invariant set such that $\mu(E) = 0$ and $\nu(E) = 1$. Given $\varepsilon > 0$, there exists $n = n(\varepsilon) \in \mathbb{N}$ and a union U_n of sets of the form $R_{i_1\cdots i_n}$ satisfying

$$\mu\left((E \setminus U_n) \cup (U_n \setminus E)\right) < \varepsilon \quad \text{and} \quad \nu\left((E \setminus U_n) \cup (U_n \setminus E)\right) < \varepsilon. \tag{54}$$

Considering the partition $\xi_n = \{R_{i_1\cdots i_n} : (i_1 \cdots i_n) \in \Sigma_n\}$ of Z we have

$$H_\mu(\xi_n) = -\sum_{i_1\cdots i_n} \mu(R_{i_1\cdots i_n}) \log \mu(R_{i_1\cdots i_n})$$

and so

$$h_\mu(T_Z) = \lim_{n\to\infty} \frac{1}{n} H_\mu(\xi_n) = \inf_{n\in\mathbb{N}} \frac{1}{n} H_\mu(\xi_n) \tag{55}$$

for any probability measure μ on Z. By (22) and (55), we obtain

$$\begin{aligned}
&n \left(h_\eta(T_Z) + \lim_{n\to+\infty} \frac{1}{n} \int_Z c_n \, d\eta \right) \\
&\le H_\eta(\xi_n) + \int_Z c_n \, d\eta + C \\
&= -\sum_{i_1\cdots i_n} \eta(R_{i_1\cdots i_n}) \log \eta(R_{i_1\cdots i_n}) + \sum_{i_1\cdots i_n} \int_{R_{i_1\cdots i_n}} c_n \, d\eta + C \\
&\le \sum_{i_1\cdots i_n} -\eta(R_{i_1\cdot i_n}) \log \eta(R_{i_1\cdots i_n}) + \sum_{i_1\cdots i_n} \int_{R_{i_1\cdots i_n}} \left[V_n(c) + c_n(x_{i_1\cdots i_n}) \right] d\eta + C
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{R_{i_1\cdots i_n}} \eta(R_{i_1\cdots i_n})\,[-\log \eta(R_{i_1\cdots i_n}) + c_n(x_{i_1\cdots i_n})] + V_n(c) + C \\
&= \sum_{R_{i_1\cdots i_n}\cap U_n=\emptyset} \eta(R_{i_1\cdots i_n})\,[-\log \eta(R_{i_1\cdots i_n}) + c_n(x_{i_1\cdots i_n})] \\
&\quad + \sum_{R_{i_1\cdots i_n}\cap U_n\neq\emptyset} \eta(R_{i_1\cdots i_n})\,[-\log \eta(R_{i_1\cdots i_n}) + c_n(x_{i_1\cdots i_n})] + V_n(c) + C
\end{aligned}
$$

for any $x_{i_1\cdots i_n} \in R_{i_1\cdots i_n}$. For example by inequality (20.3.5) in [10], we have

$$
\sum_{i=1}^{k} y_i(b_i - \log y_i) \le \sum_{i=1}^{k} y_i \log \sum_{j=1}^{k} e^{b_j} + \frac{1}{e}
$$

for any $b_i \in \mathbb{R}$ and $y_i \ge 0$ for $i = 1, \dots, k$. This gives

$$
\begin{aligned}
&n\left(h_\eta(T_Z) + \lim_{n\to+\infty} \frac{1}{n}\int_Z c_n\,d\eta\right) \\
&\le \sum_{R_{i_1\cdots i_n}\cap U_n=\emptyset} \eta(R_{i_1\cdots i_n}) \log \sum_{R_{i_1\cdots i_n}\cap U_n=\emptyset} e^{c_n(x_{i_1\cdots i_n})} + \frac{1}{e} \\
&\quad + \sum_{R_{i_1\cdots i_n}\cap U_n\neq\emptyset} \eta(R_{i_1\cdots i_n}) \log \sum_{R_{i_1\cdots i_n}\cap U_n\neq\emptyset} e^{c_n(x_{i_1\cdots i_n})} + \frac{1}{e} + V_n(c) + C \\
&\le \eta(U_n) \log \sum_{R_{i_1\cdots i_n}\cap U_n=\emptyset} e^{c_n(x_{i_1\cdots i_n})} \\
&\quad + \eta(Z\setminus U_n) \log \sum_{R_{i_1\cdots i_n}\cap U_n\neq\emptyset} e^{c_n(x_{i_1\cdots i_n})} + \frac{2}{e} + V_n(c) + C,
\end{aligned}
$$

for any $x_{i_1\cdots i_n} \in R_{i_1\cdots i_n}$. Therefore, together with the fact that μ is a Gibbs measure for c, we obtain

$$
\begin{aligned}
&n\left(h_\eta(T_Z) + \lim_{n\to+\infty} \frac{1}{n}\int_Z c_n\,d\eta - P_{T_Z}(c)\right) \\
&\le \frac{2}{e} + V_n(c) + C + \eta(U_n) \log \sum_{R_{i_1\cdots i_n}\cap U_n=\emptyset} e^{c_n(x_{i_1\cdots i_n}) - nP_{T_Z}(c)} \\
&\quad + \eta(Z\setminus U_n) \log \sum_{R_{i_1\cdots i_n}\cap U_n\neq\emptyset} e^{c_n(x_{i_1\cdots i_n}) - nP_{T_Z}(c)} \qquad (56) \\
&\le \frac{2}{e} + V_n(c) + C + \eta(U_n)\log[\tilde{K}\mu(U_n)] + \eta(Z\setminus U_n)\log[\tilde{K}\mu(Z\setminus U_n)],
\end{aligned}
$$

for some constant $\tilde{K} > 0$. Letting $\varepsilon \to 0$ in (54), we have $\eta(U_n) \to 0$ and $\mu(U_n) \to 1$ when $n \to \infty$. On the other hand, by Lemma 10, we have $\sup_{n\in\mathbb{N}} V_n(c) < \infty$ and so the right-hand side of (56) tends to $-\infty$ when $n \to \infty$. This implies that

$$
h_\eta(T_Z) + \lim_{n\to+\infty} \frac{1}{n}\int_Z c_n\,d\eta < P_{T_Z}(c)
$$

and so η is not an equilibrium measure for c. This shows that μ is the only equilibrium measure for c and $\nu = \mu$.

Finally, we show that the measure μ is mixing. Since μ is a Gibbs measure, by (51) we have

$$\varliminf_{m\to\infty} \mu(E\cap T_Z^{-m}(F)) \ge C_8\mu(E)\mu(F) \tag{57}$$

for any Borel sets $E, F \subset Z$. Proceeding as in the proof of Lemma 14, one can also show that

$$\varlimsup_{m\to\infty} \mu(E\cap T_Z^{-m}(F)) \le C_9\mu(E)\mu(F) \tag{58}$$

for any Borel sets $E, F \subset Z$ and some constant $C_9 > 0$. By (57) and (58), together for example with Lemma 20.3.5 and Proposition 20.3.6 in [10], we conclude that μ is mixing. This completes the proof of the theorem. □

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