HYPERBOLICITY OF DELAY EQUATIONS VIA COCYCLES

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Abstract. For a nonautonomous linear delay equation, we characterize the existence of an exponential dichotomy via the hyperbolicity of a cocycle over the closure of the translations of the equation, with respect to the topology of uniform convergence on compact sets. An important advantage of this approach is that the base is compact under mild additional assumptions. Moreover, we give a few applications of the equivalence of the two notions of hyperbolicity. In particular, we consider the robustness and the admissibility of the equation and we obtain stable and unstable invariant manifolds.

1. Introduction

Our main objective is to characterize the hyperbolicity of a linear delay equation (possibly nonautonomous) via the hyperbolicity of a certain cocycle over the closure of the translations of the equation, with respect to the topology of uniform convergence on compact sets. We note that this characterization is somewhat unexpected—after all we pass from a single equation to a set of equations that may contain infinitely many linearly independent functions. Moreover, it shows that a usual notion of hyperbolicity (in terms of cocycles) for a linear delay equation can be described simply in terms of the original equation, which anyways should always be the case. An important advantage of this approach is that the base becomes compact under mild additional assumptions and this compactness is indeed necessary in some applications.

1.1. **Hyperbolicity for delay equations.**

We consider the general cases of a nonautonomous linear delay equation, which causes that we need to consider general evolution families, of a delay equation satisfying the Carathéodory conditions in the theory of ordinary differential equations, and so also solutions in the sense of Carathéodory, and of a noninvertible evolution family for the linear equation other than along the unstable spaces of an exponential dichotomy.

More precisely, we consider a linear delay equation

$$v' = L(t)v_t, \tag{1}$$

where $L(t)\colon C \to \mathbb{R}^n$, for $t \in \mathbb{R}$, are bounded linear operators on the Banach space C of all continuous functions $\phi\colon [-r, 0] \to \mathbb{R}^n$ equipped with the

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supremum norm. We always assume that $t \mapsto L(t)\phi$ is measurable for each $\phi \in C$ and that

$$\sup_{t\in\mathbb{R}} \int_t^{t+1} \|L(\tau)\| \, d\tau < +\infty.$$

Under these assumptions, equation (1) determines an evolution family $T(t,s)$ on C defined by $T(t,s)\phi = v_t$ for $t \geq s$ and $\phi \in C$, where v is the unique solution of the equation on $[s-r,+\infty)$ with $v_s = \phi$. We refer the reader to the book [6] for details.

Now we consider the notion of an exponential dichotomy for equation (1), which copies verbatim the usual notion in the theory of ordinary differential equations (see Section 2 for the definition). Let $\mathcal{L}(C)$ be the set of bounded linear operators on C. One can easily verify that equation (1) has an exponential dichotomy if and only if the cocycle

$$A = A_L \colon \mathbb{R} \times \mathbb{R}_0^+ \to \mathcal{L}(C) \tag{2}$$

defined by $A_L(s,t) = T(s+t,s)$ over the flow $S_t(s) = s+t$ on $\mathbb{R}$ is hyperbolic. This classical notion of uniform hyperbolicity, essentially introduced by Perron in [11], plays an important role in a large part of the theory of differential equations and dynamical systems. Some of its consequences are the existence of topological conjugacies and of stable and unstable invariant manifolds under sufficiently small nonlinear perturbations, among many others. We refer to [6] for references and for many consequences of the presence of hyperbolicity in the context of delay equations.

1.2. **Hyperbolicity via cocycles.** But there is another common approach, particularly for nonautonomous linear equations, and not only for delay equations, which is related to the study of cocycles. For a quite early mention in the context of ordinary differential equations see for example the work of Sacker and Sell in [13]. We describe briefly this approach, for convenience already for delay equations.

For some bounded linear operators $L(t)$ as in equation (1) we define

$$L_\tau(t) = L(t+\tau) \quad \text{for } t \in \mathbb{R}.$$

Moreover, let X be the closure of the set $\{L_\tau : \tau \in \mathbb{R}\}$ with respect to the topology of uniform convergence on compact sets. For each $M \in X$ we consider the problem

$$\begin{cases} v' = M(t)v_t, \\ v_0 = \phi \end{cases}$$

and we define linear operators by $B(M,t)\phi = v_t$, for $t \geq 0$, where v is the unique solution of problem (13) on the interval $[-r,+\infty)$. Then the map

$$B = B_L \colon X \times \mathbb{R}_0^+ \to \mathcal{L}(C) \tag{3}$$

is a cocycle over the flow $S_t(M) = M_t$ on X. It induces a semiflow on $X \times C$ given by

$$\psi_t(M,\phi) = (M_t, B(M,t)\phi) \quad \text{for } t \geq 0.$$

The second approach to the study of hyperbolic behavior mimics verbatim a corresponding approach for ordinary differential equations. Namely, we can also consider the hyperbolicity of the cocycle B, following closely Magalhães in [8]. In such early works such as [13] it is already this notion of hyperbolicity that is considered, particularly since under mild additional assumptions the base X becomes compact. Indeed, one can easily verify, as a consequence of the Arzelà–Ascoli theorem, that if the map $t \mapsto L(t)$ is bounded and uniformly continuous, then the set X is compact. Incidentally, note that the base $\mathbb{R}$ for the cocycle A_L in (2) is not compact.

1.3. **Equivalence of the notions.** As noted above, it is often convenient to consider an alternative notion of hyperbolicity for the equation $v' = L(t)v_t$ that corresponds to the hyperbolicity of the cocycle B_L in (3). Of course, this situation is somewhat awkward since it seems that depending on each particular case we may need to use a specific notion of hyperbolicity, namely the hyperbolicity of any of the cocycles A_L and B_L in (2) and (3).

The main objective of our work is to show that the two notions of hyperbolicity are equivalent under a mild additional assumption. More precisely, we have the following result (see Theorem 6).

Theorem 1. *Assume that the map $t \mapsto L(t)$ is continuous and bounded. Then equation the $v' = L(t)v_t$ has an exponential dichotomy if and only if the cocycle B_L is hyperbolic.*

We emphasize that the equivalence of the two notions of hyperbolicity does not require the compactness of X. Moreover, in fact we show that if the cocycle B_L is hyperbolic, then the equation $v' = L(t)v_t$ has an exponential dichotomy even if the map $t \mapsto L(t)$ is discontinuous or unbounded.

The difficult direction in the proof of Theorem 1 is showing that if the equation $v' = L(t)v_t$ has an exponential dichotomy, then the cocycle B_L is hyperbolic. First we observe that it is simple to show that each equation

$$v' = L_\tau(t)v_t$$

also has an exponential dichotomy. A major difficulty is then to show that the same also happens for any equation $v' = M(t)v_t$ with $M \in X$ (after which it is simple to establish the hyperbolicity of the cocycle B_L). In order to explain the difficulty, we need to be a bit more detailed. Given $M \in X$, there exists a sequence $(t_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ such that for each compact set $K \subset \mathbb{R}$ one has

$$\sup_{t\in K} \|L_{t_n}(t) - M(t)\| \to 0 \quad \text{when } n \to \infty.$$

The proof proceeds in several steps:

1. We first show that
$$B(M,t) = \lim_{n\to\infty} B(L_{t_n}, t) = \lim_{n\to\infty} T(t + t_n, t_n).$$
This result is of simpler nature and amounts essentially to a careful application of Gronwall's lemma.
2. Denoting by $P(s)$ the projections associated to the exponential dichotomy of the equation $v' = L(t)v_t$, we then show that the limit
$$P_M = \lim_{n\to\infty} P(t_n) \tag{4}$$
exists. This is a delicate step since it corresponds to the convergence of the stable and unstable space, which may vary with n, particularly

since the sequence t_n may be unbounded. The argument involves obtaining explicit formulas for the difference

$$P(t + t_n) - P(t + t_m),$$

after which we are able to show that the limit in (4) exists.

3. We also need to show that the solutions of the equation $v' = M(t)v_t$ are global in the past along the image of $\mathrm{Id} - P_M$, which again is a bit delicate since the solutions are obtained as pointwise limits of solutions of the equations $v' = L_{t_n}(t)v_t$, which may be invertible only along the unstable spaces thus leaving no room for other choices.
4. The remaining steps, up to some technical aspects, are showing that the linear operators P_M are projections commuting with the dynamics and that the cocycle B_L has an inverse along the image of $\mathrm{Id} - P_M$ (which depends on the backward continuation established in the former step). After this one can finally show that there are contraction and expansion by a limiting procedure reminiscent of the proof that stable and unstable space of a hyperbolic set are continuous on the base point.

1.4. **Applications.** In Section 4 we give a few applications of the equivalence of the two notions of hyperbolicity in Theorem 1. Namely, we consider the problems of robustness and admissibility for a linear delay equation as well as the construction of stable and unstable invariant manifolds.

The robustness problem for a linear delay equation concerns the persistence of the hyperbolicity under sufficiently small linear perturbations. The study of robustness (in the context of ordinary differential equations) has a long history. In particular, it was discussed by Massera and Schäffer [9] (building on [11]), Coppel [4] and in the case of Banach spaces by Dalec'kiĭ and Kreĭn [5]. We also mention the works of Chow and Leiva [3] and Pliss and Sell [12], where the authors study the robustness property in the context of skew-product semiflows over a compact base. More recently, the robustness of an exponential dichotomy for a nonautonomous linear delay equation was considered in [2].

The admissibility problem for a linear delay equation concerns the characterization of the hyperbolicity by means of the existence and uniqueness of solutions for certain perturbations of the original equation. There is an extensive literature on the relation between admissibility and stability, also on infinite-dimensional spaces. For some of the most relevant early contributions we refer to the books by Massera and Schäffer [10] (building on their former work [9]) and by Dalec'kiĭ and Kreĭn [5]. We also refer to [7] for some early results on infinite-dimensional spaces. See [3, 12] for related results for skew-product semiflows over a compact base. In [1], it was shown that the notion of an exponential dichotomy for a nonautonomous linear delay equation can also be characterized in terms of an admissibility property for some appropriate pairs of admissible spaces.

Finally, we also address the construction of stable and unstable invariant manifolds. As in most other works dedicated to the construction of invariant manifolds (for any type of dynamics, not necessarily coming from delay equations), we obtain them as graphs after solving certain fixed point problems that must be satisfied by the global bounded solutions. We also consider the smoothness of the invariant manifolds for a large class of equations.

2. Basic notions

In this section we recall a few basic notions and results related to delay equations, exponential dichotomies and their relation to the variation of constants formula (see [6] for details). These are necessary for the remainder of the paper.

2.1. **Evolution families.** Let $|\cdot|$ be a norm on $\mathbb{R}^n$. Given $r > 0$, we denote by C the Banach space of all continuous functions $\phi\colon [-r, 0] \to \mathbb{R}^n$ equipped with the norm

$$\|\phi\| = \sup\big\{|\phi(\theta)| : -r \le \theta \le 0\big\}.$$

Given a function $v\colon [s-r, +\infty) \to \mathbb{R}^n$ and $t \ge s$, we define $v_t\colon [-r, 0] \to \mathbb{R}^n$ by $v_t(\theta) = v(t+\theta)$ for $\theta \in [-r, 0]$. Now we consider a linear delay equation

$$v' = L(t)v_t, \tag{5}$$

where $L(t)\colon C \to \mathbb{R}^n$, for $t \in \mathbb{R}$, are bounded linear operators such that $t \mapsto L(t)\phi$ is measurable for each $\phi \in C$ and

$$\sup_{t\in\mathbb{R}} \int_t^{t+1} \|L(\tau)\|\, d\tau < +\infty. \tag{6}$$

Then equation (5) determines an evolution family $T(t, s)\colon C \to C$, for $t, s \in \mathbb{R}$ with $t \ge s$, defined by

$$T(t, s)\phi = v_t$$

for $\phi \in C$, where $v = v(\cdot, s, \phi)$ is the unique solution of the equation on the interval $[s-r, +\infty)$ with $v_s = \phi$. This means that

$$T(t, t) = \mathrm{Id} \quad \text{and} \quad T(t, \tau)T(\tau, s) = T(t, s)$$

for $t \ge \tau \ge s$. It follows easily from (6) that each linear operator $T(t, s)$ is bounded. In fact we have the following stronger result.

Proposition 2. *The evolution family $T(t, s)$ associated with equation* (5) *satisfies*

$$\|T(t, s)\| \le \exp \int_s^t \|L(\tau)\|\, d\tau \quad \text{for } t \ge s.$$

Moreover, the map $(t, s, \phi) \mapsto T(t, s)\phi$ is continuous on the set

$$\big\{(t, s) \in \mathbb{R} \times \mathbb{R} : t \ge s\big\} \times C.$$

In fact, one can extend each linear operator $T(t, s)$ to a certain family of discontinuous functions. This extension is crucial for the variation of constants formula. Let C_0 be the set of all functions $\phi\colon [-r, 0] \to \mathbb{R}^n$ that are continuous on $[-r, 0)$ for which the limit

$$\phi(0^-) = \lim_{\theta\to 0^-} \phi(\theta)$$

exists. This is a Banach space when equipped with the supremum norm. We write the linear operator $L(t)\colon C \to \mathbb{R}^n$ as a Riemann–Stieltjes integral

$$L(t)\phi = \int_{-r}^{0} d\eta(t, \theta)\phi(\theta) \tag{7}$$

for some measurable map $\eta\colon \mathbb{R}\times[-r,0]\to M_n$, where M_n is the set of all $n\times n$ matrices, such that the map $\theta\mapsto\eta(t,\theta)$ has bounded variation and is left-continuous for each $t\in\mathbb{R}$. Since $\phi\in C_0$ is right-continuous, it is Riemann–Stieltjes integrable with respect to $\eta(t,\cdot)$ for each $t\in\mathbb{R}$. Hence, one can extend $L(t)$ to C_0 using (7). We continue to denote the extension by $L(t)$. Clearly,

$$\|L(t)|C\| = \|L(t)|C_0\|.$$

For $t,s\in\mathbb{R}$ with $t\ge s$, we define a map $T_0(t,s)$ on C_0 by $T_0(t,s)\phi = v_t$ for $\phi\in C_0$, where v is the unique solution of (5) on $[s-r,+\infty)$ with $v_s=\phi$.

2.2. **Hyperbolicity and cocycles.** In this section we introduce the notion of hyperbolicity. We say that equation (5) has an *exponential dichotomy* if:

1. there exist projections $P(t)\colon C\to C$, for $t\in\mathbb{R}$, such that
$$P(t)T(t,s) = T(t,s)P(s) \quad \text{for } t\ge s; \tag{8}$$
2. the linear operator
$$\overline{T}(t,s) := T(t,s)|_{\ker P(s)}\colon \ker P(s)\to\ker P(t)$$
is onto and invertible for every $t\ge s$;
3. there exist $\lambda, D>0$ such that for every $t\ge s$ we have
$$\|T(t,s)P(s)\|\le De^{-\lambda(t-s)}, \quad \|\overline{T}(s,t)Q(t)\|\le De^{-\lambda(t-s)}, \tag{9}$$
where $\overline{T}(s,t)=\overline{T}(t,s)^{-1}$ and $Q(t)=\mathrm{Id}-P(t)$.

The spaces $E(t)=P(t)C$ and $F(t)=Q(t)C$ are called, respectively, the *stable* and *unstable spaces* at time t.

It turns out that the exponential behavior extends to the space C_0 introduced in the former section. We define a linear operator $X_0\colon\mathbb{R}^n\to C_0$ by

$$(X_0p)(\theta)=\begin{cases}0 & \text{if } -r\le\theta<0,\\ p & \text{if } \theta=0\end{cases} \tag{10}$$

for $p\in\mathbb{R}^n$. For each $t\in\mathbb{R}$ we define linear operators $P_0(t),Q_0(t)\colon\mathbb{R}^n\to C_0$ by $P_0(t)=X_0-Q_0(t)$ and

$$Q_0(t)=\overline{T}(t,t+r)Q(t+r)T_0(t+r,t)X_0.$$

One can show that $P_0(t)p\in C_0\setminus C$ and $Q_0(t)p\in C$ for all $p\in\mathbb{R}^n$.

Proposition 3. *Assume that condition* (6) *holds. If equation* (5) *has an exponential dichotomy, then there exists* $\bar{D}>0$ *such that*

$$\|T_0(t,s)P_0(s)\|\le\bar{D}e^{-\lambda(t-s)}, \quad \|\overline{T}(s,t)Q_0(t)\|\le\bar{D}e^{-\lambda(t-s)}$$

for every $t,s\in\mathbb{R}$ *with* $t\ge s$.

The notion of hyperbolicity can also be described in terms of cocycles. Let $\mathcal{L}(C)$ be the set of bounded linear operators acting on C. We say that a map $A\colon V\times\mathbb{R}_0^+\to\mathcal{L}(C)$ is a *(linear) cocycle* over a flow S_t on a set V if for all $v\in V$ and $t,s\ge0$ we have:

1. $A(v,0)=\mathrm{Id}$;
2. $A(v,t+s)=A(S_s(v),t)A(v,s)$.

The cocycle A induces a semiflow on $V \times C$ given by

$$\phi_t(v,\phi) = (S_t(v), A(v,t)\phi) \quad \text{for } t \geq 0.$$

We say that the cocycle A is *hyperbolic* if:

1. there exist projections $P_v : C \to C$, for $v \in V$, such that
$$P_{S_t(v)}A(v,t) = A(v,t)P_v \quad \text{for } t \geq 0;$$
2. the linear operator
$$\bar{A}(v,t) := A(v,t)|_{\ker P_v} : \ker P_v \to \ker P_{S_t(v)}$$
is onto and invertible for every $v \in V$ and $t \geq 0$;
3. there exist $\lambda, D > 0$ such that for every $t \geq 0$ we have
$$\|A(v,t)P_v\| \leq De^{-\lambda t}, \quad \|\bar{A}(v,-t)(\mathrm{Id} - P_v)\| \leq De^{-\lambda t},$$
where $\bar{A}(v,-t) = \bar{A}(S_{-t}(v),t)^{-1}$.

In particular, one can easily verify that equation (5) has an exponential dichotomy if and only if the cocycle $A_L : \mathbb{R} \times \mathbb{R}_0^+ \to \mathcal{L}(C)$ defined by $A_L(s,t) = T(s+t,s)$ over the flow $S_t(s) = s+t$ on $\mathbb{R}$ is hyperbolic.

2.3. **Variation of constants formula.** Now we consider linear perturbations of equation (5) of the form

$$v' = (L(t) + M(t))v_t, \tag{11}$$

where $M(t) : C \to \mathbb{R}^n$, for $t \in \mathbb{R}$, are also bounded linear operators such that $t \mapsto M(t)\phi$ is measurable for each $\phi \in C$ and

$$\sup_{t\in\mathbb{R}} \int_t^{t+1} \|M(\tau)\|\, d\tau < +\infty. \tag{12}$$

By the variation of constants formula for delay equations, the unique solution v of equation (11) with $v_s = \phi \in C$ satisfies

$$v(t+\theta) = (T(t,s)\phi)(\theta) + \int_s^{t+\theta} (T_0(t,\tau)X_0M(\tau)v_\tau)(\theta)\, d\tau$$

for all $t \geq s$ and $\theta \in [-r,0]$ with $t + \theta \geq s$ (with X_0 as in (10)). It is usual to abbreviate this identity in the form

$$v_t = T(t,s)\phi + \int_s^t T_0(t,\tau)X_0M(\tau)v_\tau\, d\tau.$$

The following result details how the variation of constants formula projects onto the stable and unstable spaces.

Proposition 4. *If equation* (5) *has an exponential dichotomy, then the unique solution v of equation* (11) *with $v_s = \phi \in C$ satisfies*

$$P(t)v_t = T(t,s)P(s)\phi + \int_s^t T_0(t,\tau)P_0(\tau)M(\tau)v_\tau\, d\tau$$

and

$$Q(t)v_t = T(t,s)Q(s)\phi + \int_s^t T(t,\tau)Q_0(\tau)M(\tau)v_\tau\, d\tau$$

for all $t,s \in \mathbb{R}$ with $t \geq s$.

3. Relation between notions of hyperbolicity

In this section we introduce the main problem considered in the paper: to describe the relation between two notions of hyperbolicity for a linear delay equation that a priori could be unrelated. Namely, we consider the notion of hyperbolicity introduced in the former section and the one considered by Magalhães in [8], which corresponds to the classical one considered for example by Sacker and Sell in [13] for ordinary differential equations. Our main aim is to show that the two notions are equivalent, under a mild additional hypothesis.

3.1. **Preliminaries.** We continue to consider a linear delay equation as in (5) for some bounded linear operators $L(t)$, for $t \in \mathbb{R}$, such that $t \mapsto L(t)\phi$ is measurable for each $\phi \in C$ and property (6) holds. Now define

$$L_\tau(t) = L(t+\tau) \quad \text{for } t \in \mathbb{R}$$

and let X be the closure of the set $\{L_\tau : \tau \in \mathbb{R}\}$ with respect to the topology of uniform convergence on compact sets. For each $M \in X$ and $\phi \in C$ we consider the problem

$$\begin{cases} v' = M(t)v_t, \\ v_0 = \phi \end{cases} \tag{13}$$

and we define linear operators by $B(M,t)\phi = v_t$, for $t \geq 0$, where v is the unique solution of problem (13) on the interval $[-r, +\infty)$ (note that each $M \in X$ satisfies property (12) and so indeed each solution of equation (13) is global in the future). Then the map $B\colon X \times \mathbb{R}_0^+ \to \mathcal{L}(C)$ is a cocycle over the flow $S_t(M) = M_t$ on X. It induces a semiflow on $X \times C$ given by

$$\psi_t(M,\phi) = (M_t, B(M,t)\phi) \quad \text{for } t \geq 0.$$

In particular, indeed we have the identity

$$B(M, t+s) = B(M_s, t)B(M,s) \tag{14}$$

since the solution w of the problem

$$\begin{cases} w' = M(t+s)w_t, \\ w_0 = v_s, \end{cases} \tag{15}$$

with v as in (13), satisfies $w_t = v_{t+s}$ and so

$$\begin{aligned} B(M_s,t)B(M,s)\phi &= B(M_s,t)v_s = w_t \\ &= v_{t+s} = B(M,t+s)\phi. \end{aligned}$$

We start with a simpler observation.

Proposition 5. *If the cocycle B is hyperbolic, then equation* (5) *has an exponential dichotomy.*

Proof. Since the cocycle B is hyperbolic, the following properties hold:

1. there exist projections $P_M\colon C \to C$, for $M \in X$, such that

$$P_{M_t}B(M,t) = B(M,t)P_M \quad \text{for } t \geq 0;$$

2. the linear operator

$$\bar{B}(M,t) := B(M,t)|_{\ker P_M} \colon \ker P_M \to \ker P_{M_t}$$

is onto and invertible for every $M \in X$ and $t \geq 0$;

3. there exist $\lambda, D > 0$ such that for every $t \geq 0$ we have

$$\|B(M,t)P_M\| \leq De^{-\lambda t}, \quad \|\bar{B}(M,-t)(\mathrm{Id} - P_M)\| \leq De^{-\lambda t},$$

where $\bar{B}(M,-t) = B(M_{-t},t)^{-1}$.

Defining $P(s) = P_{L_s}$ and taking $M = L_s$ and $t = \tau - s$, we obtain the corresponding properties:

1. the projections $P(s)$ satisfy

$$P(\tau)B(L_s,\tau - s) = B(L_s,\tau - s)P(s) \quad \text{for } \tau \geq s;$$

2. the linear operator

$$\bar{B}(L_s,\tau - s) := B(L_s,\tau - s)|_{\ker P(s)} \colon \ker P(s) \to \ker P(\tau)$$

is onto and invertible for every $\tau \geq s$;

3. for every $\tau \geq s$ we have

$$\|B(L_s,\tau - s)P(s)\| \leq De^{-\lambda(\tau - s)}$$

and

$$\|\bar{B}(L_s,\tau - s)^{-1}(\mathrm{Id} - P(\tau))\| \leq De^{-\lambda(\tau - s)}.$$

Now consider the auxiliary problem

$$\begin{cases} v' = L(t+s)v_t, \\ v_0 = \phi. \end{cases} \tag{16}$$

It follows from (14) (see also (15)) that the unique solution v of problem (16) on the interval $[-r,+\infty)$ satisfies

$$T(t+s,s)\phi = v_t. \tag{17}$$

Therefore,

$$B(L_s,t) = T(t+s,s) \quad \text{and so} \quad B(L_s,\tau - s) = T(\tau,s) \tag{18}$$

for $\tau \geq s$. Hence, it follows readily from the former discussion that equation (5) has an exponential dichotomy. □

3.2. **Main result.** One can now formulate our main result, which shows that the two notions of hyperbolicity introduced above are in fact equivalent under a mild additional hypothesis.

Theorem 6. *Assume that the map $t \mapsto L(t)$ is continuous and bounded. Then equation* (5) *has an exponential dichotomy if and only if the cocycle B is hyperbolic.*

Proof. It was already shown in Proposition 5 that if the cocycle B is hyperbolic, then equation (5) has an exponential dichotomy. We divide the proof of the converse into various steps.

Step 1. An auxiliary result.

Lemma 7. *For each $s \in \mathbb{R}$, equation* (5) *has an exponential dichotomy if and only if the equation $v' = L(t+s)v_t$ has an exponential dichotomy (with the same constants λ and D).*

Proof of the lemma. First observe that by (17) we have

$$T(t+s,\tau+s)v_\tau = T(t+s,\tau+s)T(\tau+s,s)\phi = T(t+s,s)\phi = v_t.$$

In other words, the evolution family associated to the equation $v' = L(t+s)v_t$ is given by

$$U(t,\tau) = T(t+s,\tau+s).$$

Now assume that equation (5) has an exponential dichotomy. Then for every $s \in \mathbb{R}$ and $t \geq \tau$ we have

$$\|T(t+s,\tau+s)P(\tau+s)\| \leq De^{-\lambda(t+s-\tau-s)} = De^{-\lambda(t-\tau)} \tag{19}$$

and

$$\|\overline{T}(\tau+s,t+s)Q(t+s)\| \leq De^{-\lambda(t+s-\tau-s)} = De^{-\lambda(t-\tau)}. \tag{20}$$

Moreover,

$$P(t+s)U(t,\tau) = U(t,\tau)P(\tau+s)$$

and the linear operator

$$\overline{U}(t,\tau) := T(t+s,\tau+s)|_{\ker P(\tau+s)} \colon \ker P(\tau+s) \to \ker P(t+s)$$

is onto and invertible for every $t \geq \tau$. Finally, by (19) and (20) we have

$$\|U(t,\tau)P(\tau+s)\| \leq De^{-\lambda(t-\tau)}, \quad \|\overline{U}(\tau,t)Q(t+s)\| \leq De^{-\lambda(t-\tau)},$$

where $\overline{U}(\tau,t) = \overline{U}(t,\tau)^{-1}$. This shows that the equation $v' = L(t+s)v_t$ has an exponential dichotomy, with projection $P(t+s)$ at time t.

Conversely, assume that the equation $v' = L(t+s)v_t$ has an exponential dichotomy. Then there exist projections $P_s(t)$ satisfying

$$P_s(t)U(t,\tau) = U(t,\tau)P_s(\tau) \quad \text{for } t \geq \tau \tag{21}$$

and constants $\lambda, D > 0$ such that for every $t \geq \tau$ we have

$$\|U(t,\tau)P_s(\tau)\| \leq De^{-\lambda(t-\tau)} \quad \text{and} \quad \|\overline{U}(\tau,t)Q_s(t)\| \leq De^{-\lambda(t-\tau)},$$

where $Q_s(\tau) = \mathrm{Id} - P_s(\tau)$ and with $\overline{U}$ as in the notion of an exponential dichotomy. Replacing t by $t-s$ and τ by $\tau - s$, this implies that

$$\|T(t,\tau)P(\tau)\| \leq De^{-\lambda(t-\tau)} \quad \text{and} \quad \|\overline{T}(\tau,t)Q(t)\| \leq De^{-\lambda(t-\tau)},$$

taking

$$\overline{T}(\tau,t) = \overline{U}(\tau-s,t-s) \quad \text{and} \quad P(\tau) = P_s(\tau-s).$$

In other words, property (9) holds and it follows from (21) that property (8) also holds. This shows that equation (5) has an exponential dichotomy. $\square$

Step 2. Approximation of solutions. Now we assume that equation (5) has an exponential dichotomy. Given $M \in X$, there exists a sequence $(t_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ such that for each compact set $K \subset \mathbb{R}$ one has

$$\|L_{t_n} - M\|_{K,\infty} \to 0 \quad \text{when } n \to \infty, \tag{22}$$

where

$$\|L\|_{K,\infty} = \sup_{t\in K} \sup_{\psi\neq 0} \frac{\|L(t)\psi\|}{\|\psi\|}.$$

Lemma 8. *We have*

$$B(M,t) = \lim_{n\to\infty} T(t+t_n, t_n). \tag{23}$$

Proof of the lemma. The solution v^n of the problem

$$\begin{cases} v' = L_{t_n}(t)v_t, \\ v_0 = \phi \end{cases}$$

satisfies

$$v^n(t) = \phi(0) + \int_0^t L_{t_n}(\tau)v^n_\tau \, d\tau.$$

We want to compare the functions v^n with the solution v of problem (13), which satisfies

$$v(t) = \phi(0) + \int_0^t M(\tau)v_\tau \, d\tau.$$

One can easily verify that if $(t_n)_{n\in\mathbb{N}}$ has a converging subsequence to some number $r \in \mathbb{R}$, then $M = L_r$. By Lemma 7, the equation $v' = L_r(t)v_t$ has an exponential dichotomy. Hence, it suffices to consider the case when $(t_n)_{n\in\mathbb{N}}$ has no converging subsequences to real numbers. Nevertheless, the argument that follows does not need this hypothesis explicitly since nothing would change and so we will not make it.

Note that whenever $t+\theta \geq 0$ with $\theta \in [-r,0]$ we have

$$\|v^n(t+\theta) - v(t+\theta)\| \leq \int_0^{t+\theta} \|L_{t_n}(\tau)v^n_\tau - M(\tau)v_\tau\| \, d\tau.$$

Since $v^n_0 = v_0$, by Proposition 2 we obtain

$$\begin{aligned} \|v^n_t - v_t\| &\leq \int_0^t \|L_{t_n}(\tau)v^n_\tau - M(\tau)v_\tau\| \, d\tau \\ &\leq \int_0^t \|(L_{t_n}(\tau) - M(\tau))v^n_\tau\| \, d\tau + \int_0^t \|M(\tau)\| \cdot \|v^n_\tau - v_\tau\| \, d\tau \\ &\leq \int_0^t \left(\|L_{t_n}(\tau) - M(\tau)\| \cdot \|\phi\| \exp \int_0^t \|L_{t_n}(s)\| \, ds \right) d\tau \\ &\quad + \int_0^t \|M(\tau)\| \cdot \|v^n_\tau - v_\tau\| \, d\tau \end{aligned} \tag{24}$$

and so it follows from Gronwall's lemma that

$$\begin{aligned} \|v^n_t - v_t\| &\leq \int_0^t \left(\|L_{t_n}(\tau) - M(\tau)\| \cdot \|\phi\| \exp \int_0^t \|L_{t_n}(s)\| \, ds \right) d\tau \\ &\quad \times \exp \int_0^t \|M(\tau)\| \, d\tau. \end{aligned} \tag{25}$$

Now take $T > 0$ and consider the compact set $K = [0, T]$. In view of (22), the function M is continuous and so there exists $c > 0$ such that

$$\|L_{t_n}(s)\| \le c \quad \text{and} \quad \|M(s)\| \le c$$

for all $s \in [0, T]$ and $n \in \mathbb{N}$. Hence, again by (22), taking limits in (25) when $n \to \infty$ we obtain $\lim_{n\to\infty} v_t^n = v_t$ for all $t \in [0, T]$ and so also for all $t \ge 0$. By (18) this implies that

$$B(M, t) = \lim_{n\to\infty} B(L_{t_n}, t) = \lim_{n\to\infty} T(t + t_n, t_n),$$

which yields property (23). □

Step 3. Convergence of the projections and invariance. Given $t \ge 0$, let

$$P_n(t) = P(t + t_n) \quad \text{and} \quad Q_n(t) = \mathrm{Id} - P_n(t).$$

We want to show that $(P_n(t))_{n\in\mathbb{N}}$ is a Cauchy sequence for each $t \in \mathbb{R}$ and so also a convergent sequence.

First we prove an auxiliary result. Take $m, n \in \mathbb{N}$ and consider the equation

$$v' = L_{t_n}(t)v_t = \big[L_{t_m}(t) + N(t)\big]v_t,$$

where $N(t) = L_{t_n}(t) - L_{t_m}(t)$. For simplicity of the notation, we write

$$T_n(t, s) = T(t_n + t, t_n + s)$$

and we denote by $T_{n,0}(t, s)$ the extension of $T_n(t, s)$ to C_0. Moreover, let $P_{m,0}(t) = X_0 - Q_{m,0}(t)$ and

$$Q_{m,0}(t) = \overline{T}_m(t, t + r)Q_m(t + r)T_{m,0}(t + r, t)X_0,$$

where $\overline{T}_m(t, t + r)$ is the operator in the notion of hyperbolicity for the equation $v' = L_{t_m}(t)v_t$, which by Lemma 7 has an exponential dichotomy.

Lemma 9. *Given $s \in \mathbb{R}$ and $\phi \in C$, the function $v_t = T_n(t, s)P_n(s)\phi$ satisfies*

$$\begin{aligned} v_t = T_m(t, s)P_m(s)v_s &+ \int_s^t T_{m,0}(t, \tau)P_{m,0}(\tau)N(\tau)v_\tau \, d\tau \\ &- \int_t^{+\infty} \overline{T}_m(t, \tau)Q_{m,0}(\tau)N(\tau)v_\tau \, d\tau \end{aligned} \tag{26}$$

for every $t \ge s$.

Proof of the lemma. By Proposition 4 we have

$$\begin{aligned} P_m(\bar t)v_{\bar t} &= T_m(\bar t, \bar s)P_m(\bar s)v_{\bar s} + \int_{\bar s}^{\bar t} T_{m,0}(\bar t, \tau)P_{m,0}(\tau)N(\tau)v_\tau \, d\tau, \\ Q_m(\bar t)v_{\bar t} &= T_m(\bar t, \bar s)Q_m(\bar s)v_{\bar s} + \int_{\bar s}^{\bar t} T_m(\bar t, \tau)Q_{m,0}(\tau)N(\tau)v_\tau \, d\tau \end{aligned} \tag{27}$$

for $\bar t \ge \bar s \ge s$. Taking $\bar s = t$, the second identity in (27) leads to

$$Q_m(t)v_t = \overline{T}_m(t, \bar t)Q_m(\bar t)v_{\bar t} - \int_t^{\bar t} \overline{T}_m(t, \tau)Q_{m,0}(\tau)N(\tau)v_\tau \, d\tau.$$

By Lemma 7 and the second inequality in (9), since v_t is bounded we obtain

$$\|\overline{T}_m(t, \bar t)Q_m(\bar t)v_{\bar t}\| \le De^{-\lambda(\bar t - t)} \sup_{\tau \ge s} \|v_\tau\| \to 0$$

when $\bar{t} \to +\infty$. Hence,

$$Q_m(t)v_t = -\int_t^{+\infty} \overline{T}_m(t,\tau)Q_{m,0}(\tau)N(\tau)v_\tau\,d\tau,$$

which added to the first identity in (27) with $\bar{t} = t$ and $\bar{s} = s$ yields (26). □

In a similar manner to that in Lemma 9 one can prove the following result.

Lemma 10. *Given $s \in \mathbb{R}$ and $\phi \in C$, the function $v_t = \overline{T}_n(t,s)Q_n(s)\phi$ satisfies*

$$\begin{aligned} v_t = \overline{T}_m(t,s)Q_m(s)v_s &- \int_t^s \overline{T}_m(t,\tau)Q_{m,0}(\tau)N(\tau)v_\tau\,d\tau \\ &+ \int_{-\infty}^t T_{m,0}(t,\tau)P_{m,0}(\tau)N(\tau)v_\tau\,d\tau \end{aligned}$$

for every $t \le s$.

One can now establish the desired statement.

Lemma 11. *$(P_n(s))_{n\in\mathbb{N}}$ is a Cauchy sequence for each $s \in \mathbb{R}$.*

Proof of the lemma. It follows from Lemma 9 with $t = s$ that

$$P_n(s) = P_m(s)P_n(s) - \int_s^{+\infty} \overline{T}_m(s,\tau)Q_{m,0}(\tau)N(\tau)T_n(\tau,s)P_n(s)\,d\tau. \quad (28)$$

Moreover, it follows from Lemma 10 with $t = s$ that

$$Q_n(s) = Q_m(s)Q_n(s) + \int_{-\infty}^s T_{m,0}(s,\tau)P_{m,0}(\tau)N(\tau)\overline{T}_n(\tau,s)Q_n(s)\,d\tau. \quad (29)$$

Since

$$Q_n(s) - Q_m(s)Q_n(s) = P_m(s) - P_m(s)P_n(s),$$

combining (28) and (29) we obtain

$$\begin{aligned} P_m(s) - P_n(s) = &\int_s^{+\infty} \overline{T}_m(s,\tau)Q_{m,0}(\tau)N(\tau)T_n(\tau,s)P_n(s)\,d\tau \\ &+ \int_{-\infty}^s T_{m,0}(s,\tau)P_{m,0}(\tau)N(\tau)\overline{T}_n(\tau,s)Q_n(s)\,d\tau \end{aligned}$$

By Lemmas 9 and 10, the integrals on the right-hand side are well defined. Moreover, letting $c = \sup_{t\in\mathbb{R}}\|L(t)\|$ we have $\|N(t)\| \le 2c$ and it follows from (9) and Proposition 3 that

$$\begin{aligned} &\int_{s+T}^{+\infty} \|\overline{T}_m(s,\tau)Q_{m,0}(\tau)\| \cdot \|N(\tau)\| \cdot \|T_n(\tau,s)P_n(s)\|\,d\tau \\ &\le 2c\bar{D}D\int_{s+T}^{+\infty} e^{-2\lambda(\tau-s)}\,d\tau = \frac{c\bar{D}D}{\lambda}e^{-2\lambda T} \end{aligned} \quad (30)$$

and, similarly,

$$\begin{aligned} &\int_{-\infty}^{s-T} \|T_{m,0}(s,\tau)P_{m,0}(\tau)\| \cdot \|N(\tau)\| \cdot \|\overline{T}_n(\tau,s)Q_n(s)\|\,d\tau \\ &\le 2c\bar{D}D\int_{-\infty}^{s-T} e^{-2\lambda(s-\tau)}\,d\tau = \frac{c\bar{D}D}{\lambda}e^{-2\lambda T}. \end{aligned} \quad (31)$$

Given $\delta > 0$, take $T > 0$ such that

$$\frac{c\bar{D}D}{\lambda} e^{-2\lambda T} < \delta.$$

Now we consider the integrals on the intervals $[s, s+T]$ and $[s-T, s]$. Since these are compact, there exists $p \in \mathbb{N}$ (possibly depending on s) such that

$$\|L_{t_n}(t) - L_{t_m}(t)\| = \|N(t)\| < \delta$$

for every $n, m > p$ and $t \in [s-T, s+T]$. Then

$$\begin{aligned}&\int_s^{s+T} \|\overline{T}_m(s,\tau)Q_{m,0}(\tau)\| \cdot \|N(\tau)\| \cdot \|T_n(\tau,s)P_n(s)\| \, d\tau \\ &\le \delta\bar{D}D \int_s^{s+T} e^{-2\lambda(\tau-s)} \, d\tau \le \frac{\delta\bar{D}D}{2\lambda}\end{aligned}$$

and

$$\begin{aligned}&\int_{s-T}^{s} \|T_{m,0}(s,\tau)P_{m,0}(\tau)\| \cdot \|N(\tau)\| \cdot \|\overline{T}_n(\tau,s)Q_n(s)\| \, d\tau \\ &\le \delta\bar{D}D \int_{s-T}^{s} e^{-2\lambda(s-\tau)} \, d\tau \le \frac{\delta\bar{D}D}{2\lambda}.\end{aligned}$$

Together with (30) and (31) this implies that

$$\|P_m(s) - P_n(s)\| \le 2\delta + \frac{\delta\bar{D}D}{\lambda}$$

for every $n, m > p$, which shows that $(P_n(s))_{n\in\mathbb{N}}$ is a Cauchy sequence. $\square$

We note that the limit of $(P_n(s))_{n\in\mathbb{N}}$ only depends on M and not on the particular sequence $(t_n)_{n\in\mathbb{N}}$ in (22). Indeed, if $(t'_n)_{n\in\mathbb{N}}$ is another sequence such that

$$\|L_{t'_n} - M\|_{K,\infty} \to 0 \quad \text{when } n \to \infty,$$

for each compact set $K \subset \mathbb{R}$, then proceeding as in the proof of Lemma 11 replacing t_m by t'_n we find that

$$\begin{aligned}&P(s+t'_n) - P(s+t_n) \\ &= \int_s^{+\infty} \overline{T}(s+t'_n, \tau+t'_n)Q_0(\tau+t'_n)\bar{N}(\tau)T_n(\tau,s)P_n(s) \, d\tau \\ &\quad + \int_{-\infty}^{s} T_0(s+t'_n, \tau+t'_n)P_0(\tau+t'_n)\bar{N}(\tau)\overline{T}_n(\tau,s)Q_n(s) \, d\tau,\end{aligned}$$

where $\bar{N}(t) = L_{t_n}(t) - L_{t'_n}(t)$. Since the interval $[s-T, s+T]$ is compact, there exists $p \in \mathbb{N}$ such that $\|\bar{N}(t)\| < \delta$ for every $n > p$ and $t \in [s-T, s+T]$. This allows one to repeat the arguments in the lemma to find that

$$\|P(s+t'_n) - P(s+t_n)\| \le 2\delta + \frac{\delta\bar{D}D}{\lambda}$$

for every $n > p$, which shows that the limit of $(P_n(s))_{n\in\mathbb{N}}$ does not depend on the particular sequence $(t_n)_{n\in\mathbb{N}}$.

In particular, taking $s = 0$, one can define

$$P_M = \lim_{n\to\infty} P_n(0) = \lim_{n\to\infty} P(t_n). \tag{32}$$

Note that since L_{t_n} converges to M in X, the function L_{t_n+t} converges to M_t. Therefore,

$$P_{M_t} = \lim_{n\to\infty} P(t_n + t). \tag{33}$$

Since $P(t_n)^2 = P(t_n)$, we obtain

$$\begin{aligned} P_M^2 &= \lim_{n\to\infty} P(t_n) \lim_{n\to\infty} P(t_n) \\ &= \lim_{n\to\infty} P(t_n)^2 = P_M \end{aligned}$$

and so P_M is a projection. Therefore,

$$Q_M = \mathrm{Id} - P_M = \lim_{n\to\infty} Q(t_n) \tag{34}$$

is also a projection.

For the invariance, note that

$$\begin{aligned} B(M,t)P_M &= \lim_{n\to\infty} T(t+t_n, t_n)P(t_n) \\ &= \lim_{n\to\infty} P(t+t_n)T(t+t_n, t_n) = P_{M_t}B(M,t), \end{aligned} \tag{35}$$

using (33) in the last identity. In particular we obtain a map

$$B(M,t)|_{\ker P_M} \colon \ker P_M \to \ker P_{M_t}. \tag{36}$$

Step 4. Dynamics into the past and invertibility. We start with an auxiliary result.

Lemma 12. *$(\overline{T}_n(t,0)Q_n(0))_{n\in\mathbb{N}}$ is a Cauchy sequence for each $t \le 0$.*

Proof of the lemma. Note that $v_t^n = \overline{T}_n(t,0)Q_n(0)\phi$ is the unique solution of the problem

$$\begin{cases} v' = L_{t_n}(t)v_t, \\ v_0 = Q_n(0)\phi \end{cases}$$

on the interval $\mathbb{R}_0^-$ (which in view of Lemma 7 is well defined). Given $T < 0$, we have

$$v^n(t) = v^n(T) + \int_T^t L_{t_n}(\tau)v_\tau^n \, d\tau$$

for all $t \in [T,0]$. Proceeding as in (24) and (25) we obtain

$$\begin{aligned} \|v_T^n - v_T^m\| &\le \|v_t^n - v_t^m\| + \int_T^t \|L_{t_n}(\tau)v_\tau^n - L_{t_m}(\tau)v_\tau^m\| \, d\tau \\ &\le \|v_t^n - v_t^m\| \\ &\quad + \int_T^t \left(\|L_{t_n}(\tau) - L_{t_m}(\tau)\| \cdot \|\phi\| \exp \int_T^\tau \|L_{t_n}(s)\| \, ds \right) d\tau \\ &\quad + \int_T^t \|L_{t_m}(\tau)\| \cdot \|v_\tau^n - v_\tau^m\| \, d\tau. \end{aligned}$$

Taking $c = \sup_{t\in\mathbb{R}} \|L(t)\|$, this implies that

$$\|v_T^n - v_T^m\| \le \left(\|v_t^n - v_t^m\| + \int_T^t \|L_{t_n}(\tau) - L_{t_m}(\tau)\| \, d\tau \|\phi\| e^{c(t-T)} \right) e^{c(t-T)}$$

for all $m, n \in \mathbb{N}$ and $t \in [0, T]$. Since the interval $[0, T]$ is compact, given $\delta > 0$, there exists $p \in \mathbb{N}$ such that

$$\sup_{\tau \in [T,0]} \|L_{t_n}(\tau) - L_{t_m}(\tau)\| < \delta$$

for $m, n > p$. Therefore, taking $t = 0$ yields the inequality

$$\|v_T^n - v_T^m\| \le \big(\|v_0^n - v_0^m\| + \delta |T| \cdot \|\phi\| e^{c|T|}\big) e^{c|T|}$$

for $m, n > p$. Since

$$v_0^n - v_0^m = (Q_n(0) - Q_m(0))\phi$$

and the sequence $Q_n(0) = Q(t_n)$ converges, we conclude that v_T^n is a Cauchy sequence. Finally, since T is arbitrary, we conclude that $v^n(t)$ converges for each $t < 0$ when $n \to \infty$. □

It follows from Lemma 12 that one can define

$$C(M, t) = \lim_{n \to \infty} \overline{T}_n(t, 0) Q_n(0) \tag{37}$$

for $t \le 0$. We will show that

$$B(M_{-t}, t) C(M, -t) = Q_M \tag{38}$$

and

$$C(M, -t) B(M_{-t}, t) Q_{M_{-t}} = Q_{M_{-t}} \tag{39}$$

for each $t \ge 0$. This implies that the linear operator $B(M, t)|_{\ker P_M}$ in (36) is onto and invertible for each $t \ge 0$.

Since $L_{t_n - t}$ converges to M_{-t} in X, by (23) and (33), we have

$$B(M_{-t}, t) = \lim_{n \to \infty} T(t_n, t_n - t)$$

and

$$Q_{M_{-t}} = \lim_{n \to \infty} Q(t_n - t).$$

Therefore,

$$\begin{aligned} B(M_{-t}, t) C(M, -t) &= \lim_{n \to \infty} T(t_n, t_n - t) \overline{T}_n(-t, 0) Q_n(0) \\ &= \lim_{n \to \infty} T_n(0, -t) \overline{T}_n(-t, 0) Q(t_n) \\ &= \lim_{n \to \infty} Q(t_n) = Q_M \end{aligned}$$

and

$$\begin{aligned} C(M, -t) B(M_{-t}, t) Q_{M_{-t}} &= \lim_{n \to \infty} \overline{T}_n(-t, 0) Q_n(0) T(t_n, t_n - t) Q(t_n - t) \\ &= \lim_{n \to \infty} \overline{T}_n(-t, 0) Q_n(0) T_n(0, -t) Q(t_n - t) \\ &= \lim_{n \to \infty} Q(t_n - t) = Q_{M - t}. \end{aligned}$$

Step 5. Hyperbolicity of the cocycle B. Finally, we show that the cocycle B is hyperbolic. In view of (35) the first condition in the notion of hyperbolicity is satisfied, while in view of (38) and (39) the second condition is also satisfied. For the third condition, since equation (5) has an exponential dichotomy, it follows from Lemma 7 that for each $s \in \mathbb{R}$ the equation $v' = L(t+s)v_t$ also has an exponential dichotomy, with the same constants λ and D. In particular, we have

$$\|T(t+t_n, t_n)P(t_n)\| \le De^{-\lambda t}$$

and

$$\|\overline{T}(t_n - t, t_n)Q(t_n)\| \le De^{-\lambda t}$$

for $t \ge 0$. So, it follows from (23), (32), (34) and (37) that one can take limits when $n \to \infty$ in the former inequalities to obtain

$$\|B(M,t)P_M\| \le De^{-\lambda t}, \quad \|C(M,-t)Q_M\| \le De^{-\lambda t}.$$

Hence, the cocycle B is hyperbolic. This completes the proof of the theorem. □

4. Some applications

In this section we give a few applications of the equivalence of the two notions of hyperbolicity in Theorem 6. Namely, we consider the problems of robustness and of admissibility for a linear delay equation as well as the construction of stable and unstable invariant manifolds. Although these applications can be obtained in a somewhat simple manner after having all the relevant results at hands, the statements are still nontrivial.

4.1. **Robustness.** We start by considering the robustness problem for a linear delay equation concerning the persistence of the hyperbolicity under sufficiently small linear perturbations.

We consider the linear equation (5) and its linear perturbation (11), where $L(t), M(t)\colon C \to \mathbb{R}^n$, for $t \in \mathbb{R}$, are bounded linear operators such that:

1. the map $t \mapsto L(t)$ is continuous and bounded;
2. the map $t \mapsto M(t)$ is measurable for each $\phi \in C$ and (12) holds.

Let X be the closure of the set $\{L_\tau : \tau \in \mathbb{R}\}$ with respect to the topology of uniform convergence on compact sets.

Theorem 13. *Assume that equation* (5) *has an exponential dichotomy. If* $\delta := \sup_{t\in\mathbb{R}} \int_t^{t+1} \|M(\tau)\|\, d\tau$ *is sufficiently small, then for each* $\bar{L} \in X$ *and* $s \in \mathbb{R}$ *the equation*

$$v' = [\bar{L}(t) + M_s(t)]v_t \tag{40}$$

has an exponential dichotomy. Moreover:

1. *the unstable spaces for equation* (40) *are isomorphic, respectively, to the unstable spaces for equation* (5)*:*
2. *given* $\varepsilon > 0$*, if* δ *is sufficiently small, then* $\lambda - \varepsilon$ *is an exponent in the notion of an exponential dichotomy for equation* (40).

Proof. We first show that for each $\bar{L} \in X$ the equation

$$v' = \bar{L}(t)v_t \tag{41}$$

has an exponential dichotomy. When $\bar{L} = L_s$ for some $s \in \mathbb{R}$ this is the content of Lemma 7. For a general function $\bar{L} \in X$ we proceed as follows. One can show as in the proof of Proposition 5 that

$$B(\bar{L}_s, t-s) = T_{\bar{L}}(t,s)$$

is the evolution family determined by equation (41). Moreover, this equation has an exponential dichotomy with projections $P_{\bar{L}}(s) = P_{\bar{L}_s}$ and $Q_{\bar{L}}(s) = \mathrm{Id} - P_{\bar{L}}(s)$ for $s \in \mathbb{R}$. Indeed, we have

$$B(\bar{L}_s, t-s)P_{\bar{L}_s} = T_{\bar{L}}(t,s)P(s) \tag{42}$$

and

$$\bar{B}(\bar{L}_s, -t+s)Q_{\bar{L}_s} = \overline{T}_{\bar{L}}(s,t)(\mathrm{Id} - P(t)). \tag{43}$$

Since by hypothesis equation (5) has an exponential dichotomy, it follows from Theorem 6 that the cocycle B is hyperbolic. Hence, one can proceed as in the proof of Proposition 5 to conclude that:

1. the projections $P_{\bar{L}}(s)$ satisfy
$$P_{\bar{L}}(t)T_{\bar{L}}(t,s) = T_{\bar{L}}(t,s)P_{\bar{L}}(s) \quad \text{for } t \geq s;$$
2. the linear operator
$$\overline{T}_{\bar{L}}(t,s) := T_{\bar{L}}(t,s)|_{\ker P_{\bar{L}}(s)} \colon \ker P_{\bar{L}}(s) \to \ker P_{\bar{L}}(t)$$
is onto and invertible for every $t \geq s$;
3. with the same constants λ and D and in the notion of an exponential dichotomy for equation (5), for every $t \geq s$ we have
$$\|T_{\bar{L}}(t,s)P_{\bar{L}}(s)\| \leq De^{-\lambda(t-s)}, \quad \|\overline{T}_{\bar{L}}(s,t)Q(t)\| \leq De^{-\lambda(t-s)},$$
as a consequence of (42) and (43).

Now we recall a result taken from [2].

Lemma 14. *Assume that equation* (5) *has an exponential dichotomy. If* $\delta := \sup_{t\in\mathbb{R}} \int_t^{t+1} \|M(\tau)\|\, d\tau$ *is sufficiently small (depending on* λ *and* D*), then equation* (11) *has an exponential dichotomy. Moreover:*

1. *the stable and unstable spaces for equation* (11) *are isomorphic, respectively, to the stable and unstable spaces for equation* (5)*;*
2. *given* $\varepsilon > 0$*, if* δ *is sufficiently small, then* $\lambda - \varepsilon$ *is an exponent in the notion of an exponential dichotomy for equation* (11)*.*

We continue with the proof of the theorem. As shown above, for each $\bar{L} \in X$ equation (41) has an exponential dichotomy, with the same constants λ and D as in the notion of an exponential dichotomy for equation (5). Hence, one can apply Lemma 14 taking δ sufficiently small independently of $\bar{L}$ to conclude that the equation

$$v' = [\bar{L}(t) + M(t)]v_t$$

has an exponential dichotomy. Since

$$\sup_{t\in\mathbb{R}} \int_t^{t+1} \|M_s(\tau)\|\, d\tau = \sup_{t\in\mathbb{R}} \int_t^{t+1} \|M(\tau)\|\, d\tau = \delta$$

for any $s \in \mathbb{R}$, the same is true with M replaced by M_s. This establishes the first statement in the theorem.

For the remaining properties, we recall that by the construction in the proof of Theorem 6 the projections $Q_{\bar{L}}(s)$ are obtained as limits of projections $Q(t_n + s)$, where $(t_n)_{n\in\mathbb{N}}$ is some sequence such that L_{t_n} converges to $\bar{L}$ in X. Since all unstable spaces $Q(t_n + s)(C)$ have the same dimension (by property 2 in the notion of an exponential dichotomy), independently of $\bar{L}$ and s, and since and all of them are finite-dimensional (see for example [6]; this follows from the compactness of the operators $T_{\bar{L}}(t, s)$ for $t \geq s + r$), all are isomorphic. The last property in the theorem follows readily from Lemma 14 since $\bar{L}$ has also an exponential dichotomy with constants λ and D. □

4.2. **Admissibility.** Now we consider the admissibility problem for a linear delay equation concerning the characterization of the hyperbolicity by means of the existence and uniqueness of solutions for certain perturbations of the original equation.

We continue to consider the linear equation (5), where $L(t)\colon C \to \mathbb{R}^n$, for $t \in \mathbb{R}$, are bounded linear operators such that the map $t \mapsto L(t)$ is continuous and bounded. Moreover, we consider the perturbations

$$v' = L(t)v_t + g(t), \tag{44}$$

where $g\colon \mathbb{R} \to \mathbb{R}^n$ is a measurable function such that

$$|g|_Y := \sup_{t\in\mathbb{R}} \int_t^{t+1} |g(\tau)|\, d\tau < +\infty.$$

We also need to introduce appropriate Banach spaces for the notion of admissibility. Let C_b be the set of all continuous functions $v\colon \mathbb{R} \to \mathbb{R}^n$ such that

$$|v|_\infty := \sup_{t\in\mathbb{R}} |v(t)| < +\infty.$$

Note that C_b is a Banach space when equipped with the norm $|\cdot|_\infty$. Moreover, let Y be the set of all measurable functions $g\colon \mathbb{R} \to \mathbb{R}^n$ such that $|g|_Y < +\infty$, identified if they are equal almost everywhere. It is known that Y is a Banach space when equipped with the norm $|\cdot|_Y$. We say the pair of spaces (C_b, Y) is *admissible* for equation (44) if for each $g \in Y$ there exists a unique $v \in C_b$ such that

$$v_t = T(t, s)v_s + \int_s^t T_0(t, \tau)X_0 g(\tau)\, d\tau$$

for all $t, s \in \mathbb{R}$ with $t \geq s$, that is, if for each $g \in Y$ there exists a unique bounded global solution of equation (44).

Theorem 15. *If equation* (5) *has an exponential dichotomy, then the pair of spaces* (C_b, Y) *is admissible for the equation* $v' = \bar{L}(t)v_t$ *for each* $\bar{L} \in X$.

Proof. It is shown in [1] that if equation (5) has an exponential dichotomy, then the pair of spaces (C_b, Y) is admissible for this equation. On the other hand, it follows from Theorem 6 and the initial arguments in the proof of Theorem 13 that since equation (5) has an exponential dichotomy, the same happens to any equation $v' = \bar{L}(t)v_t$ with $\bar{L} \in X$. Therefore, the statement in the theorem follows readily from the result in [1]. □

In the other direction we have the following result.

Theorem 16. *If the pair of spaces* (C_b, Y) *is admissible for equation* (5), *then any equation* $v' = \bar{L}(t)v_t$ *with* $\bar{L} \in X$ *has an exponential dichotomy.*

Proof. It is proved in [1] that if the pair of spaces (C_b, Y) is admissible for a linear equation, then that equation has an exponential dichotomy. Hence, the desired result follows readily from Lemma 7. ☐

4.3. **Invariant manifolds.** Finally, we consider briefly the construction of stable and unstable invariant manifolds.

We consider perturbations of a linear delay equation (5) of the form

$$v' = L(t)v_t + g(t, v_t), \tag{45}$$

where:

1. $L(t)\colon C \to \mathbb{R}^n$, for $t \in \mathbb{R}$, are bounded linear operators such that the map $t \mapsto L(t)$ is continuous and bounded;
2. $g\colon \mathbb{R} \times C \to \mathbb{R}^n$ is a continuous map with $g(t, 0) = 0$ for all $t \in \mathbb{R}$ and there exists $\delta > 0$ such that
$$|g(t, u) - g(t, v)| \le \delta \|u - v\|$$
for every $t \in \mathbb{R}$ and $u, v \in C$.

The *stable set* V^s of equation (45) is the set of all initial conditions $(s, \phi) \in \mathbb{R} \times C$ for which the solution v_t of equation (45) with $v_s = \phi$ is defined and bounded on $[s - r, +\infty)$. One can easily verify that V^s has the following invariance property: if $(s, \phi) \in V^s$, then $(t, T(t, s)\phi) \in V^s$ for all $t \ge s$.

Now we formulate a Lipschitz stable manifold theorem. Assume that equation (5) has an exponential dichotomy and let $E(s)$ and $F(s)$ be the stable and unstable spaces at time s. Moreover, let Z be the set of all continuous functions

$$Z := \big\{(s, a) \in \mathbb{R} \times C : a \in E(s)\big\}$$

such that for each $s \in \mathbb{R}$:

1. $z(s, 0) = 0$ and $z(s, E(s)) \subset F(s)$;
2. for $a, \bar{a} \in E(s)$ we have
$$\|z(s, a) - z(s, \bar{a})\| \le \|a - \bar{a}\|.$$

The following result can be obtained repeating usual arguments for ordinary differential equations.

Proposition 17. *If equation* (5) *has an exponential dichotomy and* δ *is sufficiently small, then there exists a function* $z \in Z$ *such that*

$$V^s = \big\{(s, a + z(s, a)) : (s, a) \in \mathbb{R} \times E(s)\big\}. \tag{46}$$

In other words, the stable set V^s is the graph of the function z. We discuss briefly how this leads to the existence of further stable invariant manifolds for functions in X.

Theorem 18. *If equation* (5) *has an exponential dichotomy and* δ *is sufficiently small, then for each* $\bar{L} \in X$ *there exists a function* $z \in Z$ *such that the stable set* V^s *of the equation*

$$v' = \bar{L}(t)v_t + g(t, v_t)$$

satisfies (46). *Moreover, if* g *is of class* C^k *for some* $k \in \mathbb{N}$, $(\partial g/\partial v)(t,0) = 0$ *for all* $t \in \mathbb{R}$ *and* $\sup_{t\in\mathbb{R}}|(\partial^i g/\partial v^i)(t,v)\|$ *is sufficiently small for* $i = 1,\dots,k$, *then the function* z *is of class* C^k *in* a, $(\partial z/\partial a)(s,0) = 0$ *for all* $s \in \mathbb{R}$ *and* $\|(\partial^i z/\partial a^i)(s,a)\| \le 1$ *for all* $s \in \mathbb{R}$, $a \in E(s)$ *and* $i = 1,\dots,k$.

Proof. The first statement follows immediately from Theorem 6 and Proposition 17. The remaining properties can be obtained repeating usual arguments for ordinary differential equations. □

Analogously, one can also consider the construction of unstable invariant manifolds.

References

1. L. Barreira and C. Valls, *Hyperbolicity via admissibility in delay equations*, Indiana Univ. Math. J., to appear.
2. L. Barreira and C. Valls, *Robustness of hyperbolicity in delay equations*, J. Dynam. Differential Equations, to appear.
3. S.-N. Chow and H. Leiva, *Existence and roughness of the exponential dichotomy for skew-product semiflow in Banach spaces*, J. Differential Equations **120** (1995), 429–477.
4. W. Coppel, *Dichotomies and reducibility*, J. Differential Equations **3** (1967), 500–521.
5. Ju. Dalec'kiĭ and M. Kreĭn, *Stability of Solutions of Differential Equations in Banach Space*, Translations of Mathematical Monographs 43, American Mathematical Society, 1974.
6. J. Hale, *Theory of Functional Differential Equations*, Applied Mathematical Sciences 3, Springer, 1977.
7. B. Levitan and V. Zhikov, *Almost Periodic Functions and Differential Equations*, Cambridge University Press, 1982.
8. L. Magalhães, *The spectrum of invariant sets for dissipative semiflows*, in Dynamics of Infinite-Dimensional Systems (Lisbon, 1986), NATO Adv. Sci. Inst. Ser. F Comput. Systems Sci. 37, Springer, Berlin, 1987, pp. 161–168.
9. J. Massera and J. Schäffer, *Linear differential equations and functional analysis. I*, Ann. of Math. (2) **67** (1958), 517–573.
10. J. Massera and J. Schäffer, *Linear Differential Equations and Function Spaces*, Pure and Applied Mathematics 21, Academic Press, 1966.
11. O. Perron, *Die Stabilitätsfrage bei Differentialgleichungen*, Math. Z. **32** (1930), 703–728.
12. V. Pliss and G. Sell, *Robustness of exponential dichotomies in infinite-dimensional dynamical systems*, J. Dynam. Differential Equations **11** (1999), 471–513.
13. R. Sacker and G. Sell, *A spectral theory for linear differential systems*, J. Differential Equations **27** (1978), 320–358.

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