ALMOST ADDITIVE MULTIFRACTAL ANALYSIS FOR FLOWS

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Abstract. Building on the construction of equilibrium measures, we establish a conditional variational principle for the multifractal spectra of an almost additive family with respect to a continuous flow Φ such that the entropy map $\mu \mapsto h_\mu(\Phi)$ is upper-semicontinuous. We also show that the spectrum is continuous and that in the case of hyperbolic flows the corresponding irregular sets have full topological entropy. More generally, we consider the spectrum for the u-dimension and obtain corresponding results.

1. Introduction

1.1. **Thermodynamic formalism and dimension theory.** The thermodynamic formalism, together with its many applications, is a quite active and broad field of research. One the most basic notions is that of the topological pressure $P(\phi)$ of a continuous function ϕ with respect to a dynamical system $f\colon X \to X$. The notion was introduced by Ruelle in [22] for expansive maps and by Walters in [25] in the general case. In particular, the variational principle for the topological pressure says that

$$P(\phi) = \sup_\mu \left(h_\mu(f) + \int_X \phi \, d\mu \right),$$

where the supremum is taken over all f-invariant probability measures μ on X and where $h_\mu(f)$ is the entropy of f with respect to μ. We refer the reader to the books [15, 18, 19, 23] for details and further references.

The nonadditive thermodynamic formalism was introduced in [3] as a generalization of the (classical) thermodynamic formalism, essentially replacing the topological pressure $P(\phi)$ of a single function ϕ by the topological pressure $P(\Phi)$ of a sequence of continuous functions $\Phi = (\phi_n)_{n\in\mathbb{N}}$. Besides playing a unifying role, the nonadditive thermodynamic formalism has various nontrivial applications, particularly to the dimension theory and multifractal analysis of dynamical systems. In this respect, the discussion of the existence and uniqueness of equilibrium and Gibbs measures, among various other properties, turns out to be crucial.

Over the last decades, the dimension theory of dynamical systems steadily developed into an independent and quite active field of research (see for example the books [4, 20]). However, while the dimension theory and multifractal analysis for maps are quite developed, the corresponding theory

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for flows has experienced a slower progress. A major reason for this slower progress is that often a result for flows either follows easily from a corresponding result for maps or at least can be obtained using similar arguments, or requires substantial changes or even new ideas.

Indeed, some results for flows can be obtained in a more or less straightforward manner from those for discrete time. This is the case for example of the basic properties of the topological pressure as well as of the lower and upper capacity pressures that are conveniently introduced as general Carathéodory dimension characteristics (see [20]). Indeed, in dimension theory and multifractal analysis one needs to consider sets that need not be compact nor invariant. In particular, it is possible to obtain formulas for the lower and upper capacity topological pressures in terms of partition functions and separated sets following closely corresponding arguments for discrete time, both in the additive and in the nonadditive settings (see [8]).

On the other hand, certain results for example involving hyperbolicity and recurrence require substantial changes. In particular, in view of work of Bowen [14] and Ratner [21], any locally maximal hyperbolic set has associated Markov systems of arbitrarily small diameter. This essentially corresponds to show that one can think of the flow on the hyperbolic set as a suspension flow over a topological Markov chain. It turns out that the possible lack of additional regularity of the height function of the suspension flow may require extra care. Sometimes we may resort to the time-1 map (which is unavoidable when there is no hyperbolicity), but this causes other problems. For example, in general an invariant measure for the time-1 map need not be invariant for the flow and an ergodic measure for the flow need not be ergodic for the time-1 map. In its turn, this may require using an ergodic decomposition with respect to the time-1 map instead of with respect to the flow. The study of recurrence creates other problems, even in the presence of hyperbolicity, since it is crucial to consider appropriate distances at the level of symbolic dynamics that would not change the recurrence times. This is a delicate problem in the case of flows (see [1, 2] for details).

With all this in mind, in [8] we introduced a version of the nonadditive topological pressure for flows and we described some of its main properties, thus paving the way for a corresponding nonadditive thermodynamic formalism. In particular, we established a variational principle for the nonadditive topological pressure.

1.2. **Description of the results.** Here we consider a class of families for which it is possible not only to establish a variational principle for the topological pressure, but also to discuss the existence and uniqueness of equilibrium and Gibbs measures, as well as to describe a quite general multifractal analysis. This is the class of almost additive families: a family $(a_t)_{t>0}$ is said to be *almost additive* with respect to a flow $\Phi = (\phi_t)_{t\in\mathbb{R}}$ if there exists a constant $C > 0$ such that

$$-C + a_t + a_s \circ \phi_t \le a_{t+s} \le a_t + a_s \circ \phi_t + C$$

for every $t, s > 0$. This class occurs naturally for example in the study of nonconformal repellers.

In particular, we have the following variational principle for the topological pressure (see [9]). Let Φ be a continuous flow on a compact metric space X and let a be an almost additive family of continuous functions with tempered variation (see Section 2.1) such that

$$\sup_{t\in[0,s]} \|a_t\|_\infty < \infty \quad \text{for some } s > 0. \tag{1}$$

Then

$$P(a) = \sup_{\mu\in\mathcal{M}} \left(h_\mu(\Phi) + \lim_{t\to\infty} \frac{1}{t} \int_X a_t \, d\mu \right), \tag{2}$$

where $\mathcal{M}$ is the set of all Φ-invariant probability measures on X. Moreover, for hyperbolic flows one can establish the existence and uniqueness of the equilibrium measure of an almost additive family of continuous functions with bounded variation (see Section 5.2) as well as its Gibbs property. We say that a Φ-invariant measure μ on X is an *equilibrium measure* for the almost additive family a (with respect to the flow Φ) if the supremum in (2) is attained at μ, that is, if

$$P(a) = h_\mu(\Phi) + \lim_{t\to\infty} \frac{1}{t} \int_X a_t \, d\mu.$$

Now let Λ be a hyperbolic set for a C^1 flow Φ such that $\Phi|_\Lambda$ is topologically mixing and let a be an almost additive family of continuous functions on Λ with bounded variation satisfying (1). Then there exists a unique equilibrium measure for a (see [9] for this and other properties).

In this paper we establish a conditional variational principle for the multifractal spectra of an almost additive family, building on the construction of equilibrium measures. Here we formulate only a particular case of our main results.

Let $a = (a_t)_{t\geq 0}$ be an almost additive family of continuous functions with tempered variation satisfying (1). Given $\alpha \in \mathbb{R}$, we consider the level set

$$K_\alpha = \left\{ x \in X : \lim_{t\to\infty} \frac{a_t(x)}{t} = \alpha \right\}.$$

We also consider the function $\mathcal{E}(\alpha) \colon \mathbb{R} \to \mathbb{R}$ defined by

$$\mathcal{E}(\alpha) = h(\Phi|K_\alpha).$$

It is called the *entropy spectrum* of the family a with respect to Φ. Finally, we consider the map $\mathcal{P} \colon \mathcal{M} \to \mathbb{R}$ defined by

$$\mathcal{P}(\mu) = \lim_{t\to\infty} \frac{1}{t} \int_X a_t \, d\mu.$$

The following theorem is a particular case of our main result (see Theorem 9 that considers the more general u-dimension spectrum).

Theorem 1. *Let Φ be a continuous flow on a compact metric space X such that the map $\mu \mapsto h_\mu(\Phi)$ is upper semicontinuous and assume that for every $s \in \mathbb{R}$ the family sa has a unique equilibrium measure. If $\alpha \in \operatorname{int} \mathcal{P}(\mathcal{M})$, then $K_\alpha \neq \emptyset$ and the following properties hold:*

1. $\mathcal{E}(\alpha) = \max\{h_\mu(\Phi) : \mu \in \mathcal{M}$ *and* $\mathcal{P}(\mu) = \alpha\}$*;*
2. $\mathcal{E}(\alpha) = \min\{P(qa) - q\alpha : q \in \mathbb{R}\}$*;*

(3) *there exists an ergodic measure* $\mu_\alpha \in \mathcal{M}$ *such that* $\mathcal{P}(\mu_\alpha) = \alpha$, $\mu_\alpha(K_\alpha) = 1$ *and* $h_{\mu_\alpha}(\Phi) = \mathcal{E}(\alpha)$.

In addition, the entropy spectrum $\mathcal{E}$ *is continuous on* $\operatorname{int}\mathcal{P}(\mathcal{M})$.

To the possible extent, we follow the proof of Theorem 3 in [5] for discrete time, which in its turn is inspired by work in [11]. A key tool used in the proof is the differentiability of the topological pressure under appropriate assumptions (see Proposition 6). In particular, since sa has a unique equilibrium measure for every $s \in \mathbb{R}$, the function $s \mapsto P(sa)$ is of class C^1 on $\mathbb{R}$. Using this fact, for each level set K_α we can find a unique measure μ_α as in the statement of the theorem. Moreover, the regularity of the topological pressure allows us to establish the continuity of the spectrum.

Under the stronger assumption of the existence of a hyperbolic set Λ for a C^1 flow Φ, we also show that the irregular set

$$I(a) = \left\{ x \in \Lambda : \liminf_{t\to\infty} \frac{a_t(x)}{t} < \limsup_{t\to\infty} \frac{a_t(x)}{t} \right\}$$

has full topological entropy (see Theorem 13 for a more general statement).

Theorem 2. *Let* Λ *be a locally maximal hyperbolic set for a* C^1 *flow* Φ *such that* $\Phi|_\Lambda$ *is topologically mixing. If* a *is an almost additive family of continuous functions with bounded variation satisfying* (1) *such that* $\mathcal{P}(\nu_E) \in \operatorname{int}\mathcal{P}(\mathcal{M})$ *for the measure of maximal entropy* ν_E*, then* $h(\Phi|_{I(a)}) = h(\Phi|_\Lambda)$.

Finally, we comment briefly on the possible relation of our work to results of Bomfim and Varandas in [13] that combine large deviations and irregular sets. We continue to consider a locally maximal hyperbolic set Λ and an almost additive family of continuous functions a as in Theorem 2. Let ν be a Φ-invariant ergodic measure. By Birkhoff's ergodic theorem we have

$$\lim_{t\to\infty} \frac{a_t(x)}{t} = \lim_{t\to\infty} \frac{1}{t} \int_\Lambda a_t \, d\nu = \mathcal{P}(\nu)$$

for ν-almost every $x \in \Lambda$. For each $c > 0$ let

$$\overline{\Lambda}(a,\nu,c) = \left\{ x \in \Lambda : \limsup_{t\to\infty} \left| \frac{a_t(x)}{t} - \mathcal{P}(\nu) \right| \geq c \right\}$$

and

$$\underline{\Lambda}(a,\nu,c) = \left\{ x \in \Lambda : \liminf_{t\to\infty} \left| \frac{a_t(x)}{t} - \mathcal{P}(\nu) \right| \geq c \right\}.$$

Clearly, $\underline{\Lambda}(a,\nu,c) \subset \overline{\Lambda}(a,\nu,c)$ and one can verify that

$$I(a) = \bigcup_{c>0} \overline{\Lambda}(a,\nu,c) = \bigcup_{c>0} \underline{\Lambda}(a,\nu,c).$$

Therefore,

$$h(\Phi|_{I(a)}) \geq h(\Phi|_{\overline{\Lambda}(a,\nu,c)}) \geq h(\Phi|_{\underline{\Lambda}(a,\nu,c)})$$

for each $c > 0$. If further information about the sets $\underline{\Lambda}(a,\nu,c)$ and $\overline{\Lambda}(a,\nu,c)$ was available, one could perhaps give an alternative proof Theorem 13, thus showing that the irregular set has full topological entropy

Now we consider the case of discrete time. When Λ is repeller for a C^1 map f such that $f|_\Lambda$ is topologically mixing, it follows from more general results in [13] that

$$h(f|_\Lambda) > h(f|_{\overline{\Lambda}(a,\mu,c)}) \ge h(\Phi|_{\underline{\Lambda}(a,\mu,c)})$$

for every $c > 0$, where μ is the unique measure of maximal entropy for f. It is also shown that the functions

$$c \mapsto h(f|_{\overline{\Lambda}(a,\mu,c)}) \quad \text{and} \quad c \mapsto h(f|_{\underline{\Lambda}(a,\mu,c)})$$

are continuous, strictly decreasing and concave in a neighborhood of zero. This additional information is crucial in order to show that the irregular sets have full topological entropy. It would be quite interesting to find out whether the same happens in our setting, that is, if the maps

$$c \mapsto h(f|_{\overline{\Lambda}(a,\nu,c)}) \quad \text{and} \quad c \mapsto h(f|_{\underline{\Lambda}(a,\nu,c)})$$

share similar properties. Then one might be able to give an alternative proof of Theorem 2 and of its generalizations, although the latter should depend on appropriate cohomology assumptions for almost additive sequences.

2. Preliminaries

In this section we recall a few basic notions and results that will be used later on in the paper. This includes the notion of the topological pressure of a family of continuous functions and the notion of u-dimension.

2.1. **Topological pressure.**

Let $\Phi = (\phi_t)_{t\in\mathbb{R}}$ be a continuous flow on a compact metric space (X, d). Moreover, let $a = (a_t)_{t\ge 0}$ be a family of continuous functions $a_t \colon X \to \mathbb{R}$ with tempered variation. This means that

$$\lim_{\varepsilon\to 0} \overline{\lim_{t\to\infty}} \frac{\gamma_t(a,\varepsilon)}{t} = 0, \tag{3}$$

where

$$\gamma_t(a,\varepsilon) = \sup\bigl\{|a_t(y) - a_t(x)| : y \in B_t(x,\varepsilon) \text{ for some } x \in X\bigr\}$$

and

$$B_t(x,\varepsilon) = \bigl\{y \in X : d(\phi_s(y), \phi_s(x)) < \varepsilon \text{ for } s \in [0,t]\bigr\}.$$

Given $\varepsilon > 0$, we say that $\Gamma \subset X \times \mathbb{R}_0^+$ *covers* a set $Z \subset X$ if

$$\bigcup_{(x,t)\in\Gamma} B_t(x,\varepsilon) \supset Z$$

and we write

$$a(x,t,\varepsilon) = \sup\bigl\{a_t(y) : y \in B_t(x,\varepsilon)\bigr\} \quad \text{for } (x,t) \in \Gamma.$$

For each $Z \subset X$ and $\alpha \in \mathbb{R}$, let

$$M(Z,a,\alpha,\varepsilon) = \lim_{T\to\infty} \inf_{\Gamma} \sum_{(x,t)\in\Gamma} \exp(a(x,t,\varepsilon) - \alpha t), \tag{4}$$

with the infimum taken over all countable sets $\Gamma \subset X \times [T, +\infty)$ covering Z. When α goes from $-\infty$ to $+\infty$, the quantity in (4) jumps from $+\infty$ to 0 at a unique value and so one can define

$$P(a|_Z,\varepsilon) = \inf\bigl\{\alpha \in \mathbb{R} : M(Z,a,\alpha,\varepsilon) = 0\bigr\}.$$

Moreover, the limit

$$P(a|_Z) = \lim_{\varepsilon \to 0} P(a|_Z, \varepsilon)$$

exists and is called the *(nonadditive) topological pressure* of the family a on the set Z. It was introduced in [6] following closely a corresponding notion for discrete time (see [20]). For simplicity of the notation, we shall also write $P(a|_X) = P(a)$.

2.2. ***u*-dimension for flows.** We continue to assume that Φ is a continuous flow on a compact metric space X. Given a positive continuous function $u\colon X \to \mathbb{R}$, we consider the family of continuous functions $\bar{u} = (u_t)_{t \geq 0}$ defined by

$$u_t(x) = \int_0^t (u \circ \phi_s)(x) ds$$

for every $x \in X$ and $t > 0$. For each $Z \subset X$ and $\alpha \in \mathbb{R}$, let

$$N(Z, u, \alpha, \varepsilon) = \lim_{T \to \infty} \inf_{\Gamma} \sum_{(x,t) \in \Gamma} e^{-\alpha u(x,t,\varepsilon)},$$

with the infimum taken over all countable sets $\Gamma \subset X \times [T, +\infty)$ covering Z. Finally, we define

$$\dim_{u,\varepsilon} Z = \inf\bigl\{\alpha \in \mathbb{R} : N(Z, u, \alpha, \varepsilon) = 0\bigr\}.$$

The limit

$$\dim_u Z := \lim_{\varepsilon \to 0} \dim_{u,\varepsilon} Z$$

exists and is called the *u-dimension* of the set Z (with respect to the flow Φ). It was introduced in [10] following closely a corresponding notion for discrete time in [12]. Notice that when $u = 1$ the number $\dim_u Z$ coincides with the topological entropy $h(\Phi|_Z)$ of Φ on the set Z.

The following result shows how the topological pressure for flows is connected with the u-dimension (and follows readily from the definitions).

Proposition 3. *We have* $\dim_u Z = \alpha$*, where* α *is the unique root of the equation* $P(-\alpha\bar{u}|_Z) = 0$*.*

Given a probability measure μ on X and $\varepsilon > 0$, let

$$\dim_{u,\varepsilon} \mu = \inf\bigl\{\dim_{u,\varepsilon} Z : \mu(Z) = 1\bigr\}.$$

Then the limit

$$\dim_u \mu := \lim_{\varepsilon \to 0} \dim_{u,\varepsilon} \mu$$

exists and is called the *u-dimension* of the measure μ. Moreover, the *lower* and *upper u-pointwise dimensions* of μ at a point $x \in X$ are defined, respectively, by

$$\underline{d}_{\mu,u}(x) = \lim_{\varepsilon \to 0} \liminf_{t \to \infty} -\frac{\log \mu(B_t(x,\varepsilon))}{u(x,t,\varepsilon)}$$

and

$$\overline{d}_{\mu,u}(x) = \lim_{\varepsilon \to 0} \limsup_{t \to \infty} -\frac{\log \mu(B_t(x,\varepsilon))}{u(x,t,\varepsilon)}.$$

These were introduced in [10] following closely corresponding notions for discrete time in [12]. For an ergodic Φ-invariant probability measure μ on X, we have

$$\underline{d}_{\mu,u}(x)=\overline{d}_{\mu,u}(x)=\frac{h_\mu(\Phi)}{\int_X u d\mu} \tag{5}$$

for μ-almost every $x\in X$. These identities can be obtained as in the case of discrete time (see [12]).

2.3. **Almost additive families.** We recall that a family $a=(a_t)_{t\geq 0}$ of functions $a_t\colon X\to\mathbb{R}$ is said to be *almost additive* (with respect to a flow Φ) if there exists a constant $C>0$ such that

$$-C+a_t+a_s\circ\phi_t\leq a_{t+s}\leq a_t+a_s\circ\phi_t+C$$

for every $t,s\geq 0$. Let $\mathcal{M}_\Phi$ be the set of all Φ-invariant probability measures μ on X. We have the following result (see [9]).

Proposition 4. *Let a be an almost additive family of continuous functions with $\sup_{t\in[0,s]}\|a_t\|_\infty<\infty$ for some $s>0$. For any measure $\mu\in\mathcal{M}_\Phi$, the limit*

$$\tilde{a}(x):=\lim_{t\to\infty}\frac{a_t(x)}{t}$$

exists for μ-almost every $x\in X$. Moreover:

1. *$a_t/t\to\tilde{a}$ in $L^1(X,\mu)$ when $t\to\infty$;*
2. *$\int_X(a_t/t)d\mu\to\int_X\tilde{a}\,d\mu$ when $t\to\infty$;*
3. *the function*

$$\mathcal{M}_\Phi\ni\mu\mapsto\lim_{t\to\infty}\frac{1}{t}\int_X a_t\,d\mu$$

is continuous in the weak topology.*

We also recall a *variational principle* for the topological pressure of almost additive families that was established in [9].

Theorem 5. *Let Φ be a continuous flow on a compact metric space X and let a be an almost additive family of continuous functions with tempered variation such that $\sup_{t\in[0,s]}\|a_t\|_\infty<\infty$ for some $s>0$. Then*

$$\begin{aligned}P(a)&=\sup_{\mu\in\mathcal{M}_\Phi}\left(h_\mu(\Phi)+\int_X\lim_{t\to\infty}\frac{a_t(x)}{t}\,d\mu(x)\right)\\&=\sup_{\mu\in\mathcal{M}_\Phi}\left(h_\mu(\Phi)+\lim_{t\to\infty}\frac{1}{t}\int_X a_t\,d\mu\right).\end{aligned}$$

A measure $\nu\in\mathcal{M}_\Phi$ is said to be an *equilibrium measure* of the almost additive family a (with respect to the flow Φ) if

$$P(a)=h_\nu(\Phi)+\lim_{t\to\infty}\frac{1}{t}\int_X a_t\,d\nu.$$

3. Regularity of the topological pressure

In this section we establish some regularity properties of the topological pressure. Let $A(X)$ be the set of all almost additive families of continuous functions $a = (a_t)_{t\ge 0}$ on X with tempered variation such that

$$\sup_{t\in[0,s]} \|a_t\|_\infty < \infty \quad \text{for some } s > 0.$$

Moreover, let $E(X) \subset A(X)$ be the set of all such families with a unique equilibrium measure.

Proposition 6. *If Φ is a continuous flow on a compact metric space X and the map $\mu \mapsto h_\mu(\Phi)$ is upper semicontinuous, then the following properties hold:*

(1) *Given $a \in A(X)$, the function $s \mapsto P(a + sb)$ is differentiable at $s = 0$ for every $b \in A(X)$ if and only if $a \in E(X)$. In this case, the unique equilibrium measure μ_a of a is ergodic and*

$$\frac{d}{ds}P(a+sb)|_{s=0} = \lim_{t\to\infty} \int_X \frac{b_t}{t}\, d\mu_a. \tag{6}$$

(2) *Given an open set $I \subset \mathbb{R}$, if $a + sb \in E(X)$ for every $s \in I$, then the function $s \mapsto P(a + sb)$ is of class C^1 on I.*

Proof. We follow the proof of Proposition 4 in [5] that considers the case of discrete time (see also [18]). It is shown in [9] that if the map $\mu \mapsto h_\mu(\Phi)$ is upper semicontinuous, then any family in $A(X)$ has at least one equilibrium measure. Take $s \in \mathbb{R}$ and $a, b \in A(X)$. Then $a + sb \in A(X)$ and so there exists an equilibrium measure $\bar\mu_s$ for $a + sb$. By Theorem 5, we obtain

$$\begin{aligned} P(a+sb) - P(a) &\ge h_{\bar\mu_0}(\Phi) + \lim_{t\to\infty}\int_X \frac{a_t + sb_t}{t}\, d\bar\mu_0 - P(a) \\ &= s \lim_{t\to\infty}\int_X \frac{b_t}{t}\, d\bar\mu_0 \end{aligned}$$

and

$$\begin{aligned} P(a+sb) - P(a) &\le P(a+sb) - h_{\bar\mu_s}(\Phi) - \lim_{t\to\infty}\int_X \frac{a_t}{t}\, d\bar\mu_s \\ &= s \lim_{t\to\infty}\int_X \frac{b_t}{t}\, d\bar\mu_s. \end{aligned}$$

Therefore,

$$\lim_{t\to\infty}\int_X \frac{b_t}{t}\, d\bar\mu_0 \le \frac{P(a+sb)-P(a)}{s} \le \lim_{t\to\infty}\int_X \frac{b_t}{t}\, d\bar\mu_s \tag{7}$$

for $s > 0$ and

$$\lim_{t\to\infty}\int_X \frac{b_t}{t}\, d\bar\mu_0 \ge \frac{P(a+sb)-P(a)}{s} \ge \lim_{t\to\infty}\int_X \frac{b_t}{t}\, d\bar\mu_s \tag{8}$$

for $s < 0$.

Assume that the function $s \mapsto P(a + sb)$ is differentiable at $s = 0$ for every $b \in A(X)$. Moreover, assume that μ_a and ν_a are two equilibrium measures for a. For a continuous function $c\colon X \to \mathbb{R}$, we consider the family of functions given by

$$c_t(x) = \int_0^t (c \circ \phi_s)(x)\, ds$$

for $t > 0$ and $x \in X$. It follows from Birkhoff's ergodic theorem that

$$\lim_{t\to\infty} \int_X \frac{c_t}{t}\, d\mu = \int_X c\, d\mu$$

for each $\mu \in \mathcal{M}_\Phi$. Since the map $s \mapsto P(a + sb)$ is differentiable at $s = 0$, it follows from (7) and (8) that

$$\begin{aligned}\int_X c\, d\mu_a &= \lim_{t\to\infty} \int_X \frac{c_t}{t}\, d\mu_a \\ &= \lim_{s\to 0} \frac{P(a+sc) - P(a)}{s} \\ &= \lim_{t\to\infty} \int_X \frac{c_t}{t}\, d\nu_a = \int_X c\, d\nu_a.\end{aligned}$$

The arbitrariness of c guarantees that $\mu_a = \nu_a$ and so $a \in E(X)$.

We continue with an auxiliary result.

Lemma 7. *If $\bar\mu_{s_n} \to \mu$ when $n \to \infty$ for some sequence $s_n \to 0$, then μ is an equilibrium measure for a.*

Proof of the lemma. It follows from Theorem 5 that

$$P(a) \ge h_\mu(\Phi) + \lim_{t\to\infty} \frac{1}{t} \int_X a_t\, d\mu. \tag{9}$$

Moreover, the map

$$\nu \mapsto h_\nu(\Phi) + \lim_{t\to\infty} \frac{1}{t} \int_X a_t\, d\nu$$

is upper semicontinuous (see [9]). Therefore,

$$\begin{aligned}&h_\mu(\Phi) + \lim_{t\to\infty} \frac{1}{t} \int_X a_t\, d\mu \\ &\ge \limsup_{n\to\infty} \left(h_{\bar\mu_{s_n}}(\Phi) + \lim_{t\to\infty} \frac{1}{t} \int_X a_t\, d\bar\mu_{s_n} \right) \\ &= \limsup_{n\to\infty} \left(h_{\bar\mu_{s_n}}(\Phi) + \lim_{t\to\infty} \frac{1}{t} \int_X (a_t + s_n b_t)\, d\bar\mu_{s_n} - s_n \lim_{t\to\infty} \frac{1}{t} \int_X b_t\, d\bar\mu_{s_n} \right) \\ &= \limsup_{n\to\infty} \left(P(a + s_n b) - s_n \lim_{t\to\infty} \frac{1}{t} \int_X b_t\, d\bar\mu_{s_n} \right).\end{aligned} \tag{10}$$

On the other hand, we obtain

$$
\begin{aligned}
P(a+s_nb) &= \sup_{\mu\in\mathcal{M}_\Phi}\left(h_\mu(\Phi)+\lim_{t\to\infty}\frac{1}{t}\int_X (a_t+s_nb_t)\,d\mu\right)\\
&\ge \sup_{\mu\in\mathcal{M}_\Phi}\left(h_\mu(\Phi)+\lim_{t\to\infty}\frac{1}{t}\int_X a_t\,d\mu\right)\\
&\quad - \sup_{\mu\in\mathcal{M}_\Phi}\left(-s_n\lim_{t\to\infty}\frac{1}{t}\int_X b_t\,d\mu\right)\\
&= P(a)-\sup_{\mu\in\mathcal{M}_\Phi}\left(-s_n\lim_{t\to\infty}\frac{1}{t}\int_X b_t\,d\mu\right).
\end{aligned}
\tag{11}
$$

Now we observe that since the family b is almost additive, we have

$$\|b_{[t]}\|_\infty \le [t](\|b_1\|_\infty + C) \tag{12}$$

for every $t>0$, where $[\cdot]$ denotes the integer part. Hence, for every $\mu\in\mathcal{M}_\Phi$ and $t>0$ we have

$$\left|\int_X \frac{b_{[t]}}{[t]}\,d\mu\right| \le \frac{\|b_{[t]}\|_\infty}{[t]} \le \|b_1\|_\infty + C =: D.$$

This implies that

$$-s_n\lim_{t\to\infty}\frac{1}{t}\int_X b_t\,d\mu = -s_n\lim_{t\to\infty}\frac{1}{[t]}\int_X b_{[t]}\,d\mu \le |s_n|D.$$

Similarly,

$$-s_n\lim_{t\to\infty}\frac{1}{t}\int_X b_t\,d\bar\mu_{s_n} \ge -|s_n|D$$

for every $n\in\mathbb{N}$, and it follows from (10) and (11) that

$$h_\mu(\Phi)+\lim_{t\to\infty}\frac{1}{t}\int_X a_t\,d\mu \ge P(a)-2|s_n|D.$$

Letting $s_n\to 0$ gives

$$h_\mu(\Phi)+\lim_{t\to\infty}\frac{1}{t}\int_X a_t\,d\mu \ge P(a)$$

and it follows from (9) that

$$h_\mu(\Phi)+\lim_{t\to\infty}\frac{1}{t}\int_X a_t\,d\mu = P(a).$$

In other words, μ is an equilibrium measure for a. □

We proceed with the proof of the proposition. Take $a\in E(X)$, $b\in A(X)$ and $s\in\mathbb{R}$. Let $\bar\mu_s$ be an equilibrium measure for $a+sb$ and let μ_a be the unique equilibrium measure for a. Since $a\in E(X)$, it follows from Lemma 7 that $\bar\mu_s\to\mu_a$ when $s\to 0$. By (7) and (8), we obtain

$$\frac{d}{ds}P(a+sb)|_{s=0} = \lim_{s\to0}\frac{P(a+sb)-P(a)}{s} = \lim_{t\to\infty}\frac{1}{t}\int_X b_t\,d\mu_a,$$

which establishes identity (6). Now we show that μ_a is ergodic. Otherwise it would exist a Φ-invariant measurable set $Y \subset X$ with $0 < \mu_a(Y) < 1$. We consider two Φ-invariant probability measures ν_1 and ν_2 defined by

$$\nu_1(A) = \frac{\mu_a(A \cap Y)}{\mu_a(Y)} \quad \text{and} \quad \nu_2(A) = \frac{\mu_a(A \cap (X \setminus Y))}{\mu_a(X \setminus Y)}$$

for every measurable set $A \subset X$. Then

$$h_{\mu_a}(\Phi) = h_{\mu_a}(\phi_1) = \mu_a(Y) h_{\nu_1}(\phi_1) + \mu_a(X \setminus Y) h_{\nu_2}(\phi_1)$$

and

$$\int_Y a_t \, d\mu_a = \mu_a(Y) \int_X a_t \, d\nu_1, \quad \int_{X \setminus Y} a_t \, d\mu_a = \mu_a(X \setminus Y) \int_X a_t \, d\nu_2$$

for every $t > 0$. We obtain

$$\begin{aligned} P(a) &= h_{\mu_a}(\Phi) + \lim_{t\to\infty} \frac{1}{t} \int_X a_t \, d\mu_a \\ &= \mu_a(Y) h_{\nu_1}(\Phi) + \mu_a(X \setminus Y) h_{\nu_2}(\Phi) \\ &\quad + \mu_a(Y) \lim_{t\to\infty} \frac{1}{t} \int_X a_t \, d\nu_1 + \mu_a(X \setminus Y) \lim_{t\to\infty} \frac{1}{t} \int_X a_t \, d\nu_2 \\ &\le \max_{j=1,2} \left\{ h_{\nu_j}(\Phi) + \lim_{t\to\infty} \frac{1}{t} \int_X a_t \, d\nu_j \right\} \le P(a). \end{aligned}$$

This implies that either ν_1 or ν_2 is an equilibrium measure for a and so it must be equal to μ_a. By construction, $\nu_j \neq \mu_a$ for $i = 1, 2$. Therefore, μ_a must be ergodic, which establishes the first statement in the proposition.

Now we prove the second statement. Let $I \subset \mathbb{R}$ be an open set such that $a + sb \in E(X)$ for every $s \in I$. For each $\sigma \in I$, we have

$$\begin{aligned} D(\sigma) :&= \frac{d}{ds} P(a + sb)|_{s=\sigma} \\ &= \frac{d}{ds} P\big(a + \sigma b + (s - \sigma) b\big)|_{s-\sigma=0} \\ &= \lim_{t\to\infty} \frac{1}{t} \int_X b_t \, d\mu_{a+\sigma b} \end{aligned} \tag{13}$$

and so the function $s \mapsto P(a + sb)$ is differentiable on I. For a sequence $(\sigma_n)_{n\in\mathbb{N}}$ in I such that $\sigma_n \to \sigma$ when $n \to \infty$, it follows from (13) that

$$D(\sigma_n) = \lim_{t\to\infty} \frac{1}{t} \int_X b_t \, d\mu_{a+\sigma_n b} = \lim_{t\to\infty} \frac{1}{t} \int_X b_t \, d\mu_{a+\sigma b+(\sigma_n-\sigma)b}.$$

On the other hand, by item (3) in Proposition 4, the map

$$F(\mu) = \lim_{t\to\infty} \frac{1}{t} \int_X b_t \, d\mu$$

is continuous on $\mathcal{M}_\Phi$ and since $\sigma_n \to \sigma$, it follows from Lemma 7 that

$$\lim_{n\to\infty} F(\mu_{a+\sigma b+(\sigma_n-\sigma)b}) = F(\mu_{a+\sigma b}).$$

Therefore,

$$\begin{aligned}\lim_{n\to\infty} D(\sigma_n) &= \lim_{n\to\infty}\lim_{t\to\infty}\frac{1}{t}\int_X b_t\,d\mu_{a+\sigma b+(\sigma_n-\sigma)b}\\ &= \lim_{n\to\infty} F(\mu_{a+\sigma b+(\sigma_n-\sigma)b})\\ &= F(\mu_{a+\sigma b}) = \lim_{t\to\infty}\frac{1}{t}\int_X b_t\,d\mu_{a+\sigma b} = D(\sigma)\end{aligned}$$

and so the derivative is continuous. □

4. Multifractal analysis

In this section we obtain an almost additive multifractal analysis for flows.

Let $a = (a_t)_{t\geq 0}$ and $b = (b_t)_{t\geq 0}$ be almost additive families of continuous functions in $A(X)$ (see Section 3). We assume that

$$\liminf_{t\to\infty}\frac{b_t(x)}{t} > 0 \quad\text{and}\quad b_t(x) > 0 \tag{14}$$

for every $x \in X$ and $t \geq 0$. Given $\alpha \in \mathbb{R}$, we consider the level set

$$K_\alpha = \left\{x \in X : \lim_{t\to\infty}\frac{a_t(x)}{b_t(x)} = \alpha\right\}. \tag{15}$$

Proposition 8. *For every $x \in X$ and $s \in \mathbb{R}$ we have*

$$\limsup_{t\to\infty}\frac{a_t(x)}{b_t(x)} = \limsup_{t\to\infty}\frac{a_t(\phi_s(x))}{b_t(\phi_s(x))} \tag{16}$$

and

$$\liminf_{t\to\infty}\frac{a_t(x)}{b_t(x)} = \liminf_{t\to\infty}\frac{a_t(\phi_s(x))}{b_t(\phi_s(x))}. \tag{17}$$

Proof. Since a and b are almost additive families with respect to Φ, there exists $C > 0$ such that

$$-C + a_{t-s}(\phi_s(x)) + a_s(x) \leq a_t(x) \leq a_s(x) + a_{t-s}(\phi_s(x)) + C$$

and

$$-C + b_{t-s}(\phi_s(x)) + b_s(x) \leq b_t(x) \leq b_s(x) + b_{t-s}(\phi_s(x)) + C$$

for all $t \geq s \geq 0$ and $x \in X$. Hence,

$$|a_t(x) - a_{t-s}(\phi_s(x))| \leq \sup_{t\in[0,s]}\|a_t\|_\infty + C =: r_1 < \infty \tag{18}$$

and

$$|b_t(x) - b_{t-s}(\phi_s(x))| \leq \sup_{t\in[0,s]}\|b_t\|_\infty + C =: r_2 < \infty. \tag{19}$$

By the first inequality in (14), we have $\liminf_{t\to\infty} b_t(x) = +\infty$ for all $x \in X$. Together with (18) and (19), this implies that

$$\frac{a_t(x)}{b_t(x)} \leq \frac{a_{t-s}(\phi_s(x)) + r_1}{b_{t-s}(\phi_s(x)) - r_2} = \frac{a_{t-s}(\phi_s(x))}{b_{t-s}(\phi_s(x)) - r_2} + \frac{r_1}{b_{t-s}(\phi_s(x)) - r_2}$$

and

$$\frac{a_t(x)}{b_t(x)} \geq \frac{a_{t-s}(\phi_s(x)) - r_1}{b_{t-s}(\phi_s(x)) + r_2} = \frac{a_{t-s}(\phi_s(x))}{b_{t-s}(\phi_s(x)) + r_2} - \frac{r_1}{b_{t-s}(\phi_s(x)) + r_2}$$

for any sufficiently large $t > 0$. Letting $t \to \infty$ we obtain

$$\limsup_{t\to\infty} \frac{a_t(x)}{b_t(x)} = \limsup_{t\to\infty} \frac{a_{t-s}(\phi_s(x))}{b_{t-s}(\phi_s(x))} = \limsup_{t\to\infty} \frac{a_t(\phi_s(x))}{b_t(\phi_s(x))} \tag{20}$$

and

$$\liminf_{t\to\infty} \frac{a_t(x)}{b_t(x)} = \liminf_{t\to\infty} \frac{a_{t-s}(\phi_s(x))}{b_{t-s}(\phi_s(x))} = \liminf_{t\to\infty} \frac{a_t(\phi_s(x))}{b_t(\phi_s(x))}. \tag{21}$$

Now take $s < 0$ and let $y = \phi_s(x)$. By (20) and (21) with x and s replaced, respectively, by y and $-s$, we have

$$\limsup_{t\to\infty} \frac{a_t(y)}{b_t(y)} = \limsup_{t\to\infty} \frac{a_t(\phi_{-s}(y))}{b_t(\phi_{-s}(y))}$$

and

$$\liminf_{t\to\infty} \frac{a_t(y)}{b_t(y)} = \liminf_{t\to\infty} \frac{a_t(\phi_{-s}(y))}{b_t(\phi_{-s}(y))}.$$

Since $\phi_{-s}(y) = x$, these identities yield (16) and (17) for $s > 0$. □

It follows readily from Proposition 8 that $\phi_s(K_\alpha) = K_\alpha$ for every $s \in \mathbb{R}$. In other words, the level sets are Φ-invariant.

Now we consider the function $\mathcal{F}_u(\alpha)\colon \mathbb{R} \to \mathbb{R}$ defined by

$$\mathcal{F}_u(\alpha) = \dim_u K_\alpha.$$

It is called the *u-dimension spectrum* of the pair (a, b) with respect to Φ. We also consider the map $\mathcal{P}\colon \mathcal{M}_\Phi \to \mathbb{R}$ defined by

$$\mathcal{P}(\mu) = \frac{\lim_{t\to\infty} \frac{1}{t} \int_X a_t \, d\mu}{\lim_{t\to\infty} \frac{1}{t} \int_X b_t \, d\mu} = \lim_{t\to\infty} \frac{\int_X a_t \, d\mu}{\int_X b_t \, d\mu}.$$

By Proposition 4, this map is continuous and since $\mathcal{M}_\Phi$ is compact and connected, the image $\mathcal{P}(\mathcal{M}_\Phi)$ is also compact and connected.

The following theorem is our main result.

Theorem 9. *Let Φ be a continuous flow on a compact metric space X such that the map $\mu \mapsto h_\mu(\Phi)$ is upper semicontinuous and assume that*

$$\operatorname{span}\{a, b, \bar{u}\} \subset E(X).$$

If $\alpha \notin \mathcal{P}(\mathcal{M}_\Phi)$, then $K_\alpha = \emptyset$. Moreover, if $\alpha \in \operatorname{int} \mathcal{P}(\mathcal{M}_\Phi)$, then $K_\alpha \neq \emptyset$ and the following properties hold:

(1) *$\mathcal{F}_u(\alpha)$ satisfies the variational principle*

$$\mathcal{F}_u(\alpha) = \max\left\{ \frac{h_\mu(\Phi)}{\int_X u \, d\mu} : \mu \in \mathcal{M}_\Phi \text{ and } \mathcal{P}(\mu) = \alpha \right\}; \tag{22}$$

(2) *$\mathcal{F}_u(\alpha) = \min\{S_u(\alpha, q) : q \in \mathbb{R}\}$, where $S_u(\alpha, q)$ is the unique real number such that*

$$P\big(q(a - \alpha b) - S_u(\alpha, q)\bar{u}\big) = 0;$$

(3) *there exists an ergodic measure $\mu_\alpha \in \mathcal{M}_\Phi$ such that $\mathcal{P}(\mu_\alpha) = \alpha$, $\mu_\alpha(K_\alpha) = 1$ and*

$$\dim_u \mu_\alpha = \frac{h_{\mu_\alpha}(\Phi)}{\int_X u \, d\mu_\alpha} = \mathcal{F}_u(\alpha). \tag{23}$$

In addition, the spectrum $\mathcal{F}_u$ is continuous on $\operatorname{int} \mathcal{P}(\mathcal{M}_\Phi)$.

Proof. We follow the proof of Theorem 3 in [5] that considers the case of discrete time.

Lemma 10. *If* $\alpha \in \mathcal{P}(\mathcal{M}_\Phi)$, *then*

$$\inf_{q\in\mathbb{R}} P\big(q(a-\alpha b) - \mathcal{F}_u(\alpha)\bar{u}\big) \geq 0.$$

Proof of the lemma. Take $\delta > 0$ and $N \in \mathbb{N}$. We define the set

$$D_{\delta,N} = \big\{x \in X : |a_t(x) - \alpha b_t(x)| < \delta t \text{ for } t \geq N\big\}.$$

Write $r = \sup_{s\in[0,1]} \|b_s\|_\infty + C$. Note that

$$\lim_{t\to\infty} \frac{a_t(x)}{b_t(x)} = \alpha \tag{24}$$

for $x \in K_\alpha$ and given $\delta > 0$, there exists $N \in \mathbb{N}$ such that

$$\left|\frac{a_t(x)}{b_t(x)} - \alpha\right| < \frac{\delta}{2r} \tag{25}$$

for every $t > N$. Since b is almost additive, it follows from (12) that

$$\begin{aligned} b_t(x) &\leq b_{[t]}(x) + b_{t-[t]}(\phi_{[t]}(x)) + C \\ &\leq [t](\|b_1\|_\infty + C) + r \\ &\leq [t]r + r \leq 2tr \end{aligned} \tag{26}$$

for every $x \in X$ and $t > 0$. By (25) and (26), we obtain

$$|a_t(x) - \alpha b_t(x)| < \frac{\delta |b_t(x)|}{2r} \leq \delta t.$$

Therefore $x \in D_{\delta,N}$ and

$$K_\alpha \subset \bigcap_{\delta>0} \bigcup_{N\in\mathbb{N}} D_{\delta,N}.$$

By property (3), for each $\delta > 0$ we have

$$\overline{\lim_{t\to\infty}} \frac{\gamma_t(a,\varepsilon)}{t} < \delta$$

for any sufficiently small $\varepsilon > 0$. Therefore, $\gamma_t(a,\varepsilon)/t < \delta$ for large t and so

$$|a_t(z) - a_t(y)| < \delta t$$

for every $z, y \in B_t(\overline{x}, \varepsilon)$. This implies that

$$|a(\overline{x}, t, \varepsilon) - a_t(y)| \leq \sup_{z\in B_t(\overline{x},\varepsilon)} |a_t(z) - a_t(y)| \leq \delta t$$

and, similarly,

$$|b(\overline{x}, t, \varepsilon) - b_t(y)| \leq \delta t,$$

for $\overline{x} \in X$ and $y \in B_t(\overline{x}, \varepsilon)$. Given $q \in \mathbb{R}$ and $y \in B_t(\overline{x}, \varepsilon) \cap D_{\delta,N}$, we have

$$\begin{aligned} -\left[q(a-\alpha b)\right](\overline{x}, t, \varepsilon) &\leq |q| \cdot |a(\overline{x}, t, \varepsilon) - \alpha b(\overline{x}, t, \varepsilon)| \\ &\leq |q| \cdot |a(\overline{x}, t, \varepsilon) - a_t(y)| + |q| \cdot |a_t(y) - \alpha b_t(y)| \\ &\quad + |q| \cdot |\alpha b(\overline{x}, t, \varepsilon) - \alpha b_t(y)| \\ &\leq |q|\delta t + |q|\delta t + |q| \cdot |\alpha|\delta t = (2 + |\alpha|)|q|\delta t. \end{aligned}$$

Letting $v = -\mathcal{F}_u(\alpha)\bar{u}$, we obtain

$$\begin{aligned} v(\overline{x}, t, \varepsilon) - \lambda t &= [v + q(a - \alpha b)]\,(\overline{x}, t, \varepsilon) - [q(a - \alpha b)]\,(\overline{x}, t, \varepsilon) - \lambda t \\ &\leq [v + q(a - \alpha b)]\,(\overline{x}, t, \varepsilon) - [\lambda - (2 + |\alpha|)|q|\delta]\,t \end{aligned}$$

for every $\lambda \in \mathbb{R}$. Consider a set $\Gamma \subset X \times [N, +\infty)$ covering $D_{\delta,N}$ such that $B_t(x, \varepsilon) \cap D_{\delta,N} \neq \emptyset$ for every $(x, t) \in \Gamma$. Then

$$\begin{aligned} &\sum_{(x,t)\in\Gamma} \exp\left(v(\overline{x}, t, \varepsilon) - \lambda t\right) \\ &\leq \sum_{(x,t)\in\Gamma} \exp\left([v + q(a - \alpha b)]\,(\overline{x}, t, \varepsilon) - [\lambda - (2 + |\alpha|)|q|\delta]\,t\right). \end{aligned}$$

Taking the infimum over all countable sets $\Gamma \subset X \times [N, +\infty)$ covering $D_{\delta,N}$ and letting $N \to \infty$, we obtain

$$M(D_{\delta,N}, v, \lambda, \varepsilon) \leq M\big(D_{\delta,N}, v + q(a - \alpha b), \lambda - (2 + |\alpha|)|q|\delta, \varepsilon\big).$$

Therefore,

$$P(v|_{D_{\delta,N}}, \varepsilon) \leq P([v + q(a - \alpha b)]\,|_{D_{\delta,N}}, \varepsilon) + (2 + |\alpha|)|q|\delta$$

and letting $\varepsilon \to 0$ we conclude that

$$P(v|_{D_{\delta,N}}) \leq P([v + q(a - \alpha b)]\,|_{D_{\delta,N}}) + (2 + |\alpha|)|q|\delta$$

for $q \in \mathbb{R}$ and $\delta > 0$. Now we observe that by Proposition 3, $P(v|_{K_\alpha}) = 0$. On the other hand, if $Z_1 \subset Z_2$, then $P(c|_{Z_1}) \leq P(c|_{Z_2})$ and for any countable family of sets Z_j, for $j \in \mathbb{N}$, we have

$$P(c|_{\bigcup_{j\in\mathbb{N}}} Z_j) = \sup_{j\in\mathbb{N}} P(c|_{Z_j}),$$

for any family of continuous functions $c = (c_t)_{t\geq 0}$ with tempered variation (see [8]). Hence,

$$\begin{aligned} 0 = P(v|_{K_\alpha}) \leq P(v|_{\bigcup_{N\in\mathbb{N}} D_{\delta,N}}) &= \sup_{N\in\mathbb{N}} P(v|_{D_{\delta,N}}) \\ &\leq \sup_{N\in\mathbb{N}} P([v + q(a - \alpha b)]\,|_{D_{\delta,N}}) + (2 + |\alpha|)|q|\delta \\ &\leq P(v + q(a - \alpha b)) + (2 + |\alpha|)|q|\delta \end{aligned}$$

for every $\delta > 0$ and $q \in \mathbb{R}$. Since δ is arbitrary, we finally obtain

$$P\big(q(a - \alpha b) - \mathcal{F}_u(\alpha)\bar{u}\big) \geq 0$$

for every $q \in \mathbb{R}$. ☐

Lemma 11. *If* $\alpha \in \operatorname{int} \mathcal{P}(\mathcal{M}_\Phi)$, *then*

$$\min_{q\in\mathbb{R}} P\big(q(a - \alpha b) - \mathcal{F}_u(\alpha)\bar{u}\big) = 0$$

and there exists an ergodic equilibrium measure $\mu_\alpha \in \mathcal{M}_\Phi$ *with* $\mathcal{P}(\mu_\alpha) = \alpha$, $\mu_\alpha(K_\alpha) = 1$ *and*

$$\dim_u \mu_\alpha = \frac{h_{\mu_\alpha}(\Phi)}{\int_X u\, d\mu_\alpha} = \mathcal{F}_u(\alpha).$$

Proof of the lemma. Denote by r the distance from α to $\mathbb{R} \setminus \mathcal{P}(\mathcal{M}_\Phi)$ and let

$$L(q) = P\big(q(a - \alpha b) - \mathcal{F}_u(\alpha)\bar{u}\big)$$

for every $q \in \mathbb{R}$. For $\beta \in \mathbb{R}$ with $\beta = \alpha + r(\operatorname{sgn} q)/2$, we have

$$|\beta - \alpha| = \left|\frac{r(\operatorname{sgn} q)}{2}\right| = \frac{r}{2} < r,$$

which implies that $\beta \in \mathcal{P}(\mathcal{M}_\Phi)$. This implies that there exists $\mu \in \mathcal{M}_\Phi$ satisfying

$$\mathcal{P}(\mu) = \frac{\lim_{t\to\infty} \frac{1}{t} \int_X a_t \, d\mu}{\lim_{t\to\infty} \frac{1}{t} \int_X b_t \, d\mu} = \beta,$$

that is,

$$\lim_{t\to\infty} \frac{1}{t} \int_X a_t \, d\mu = \lim_{t\to\infty} \frac{1}{t} \int_X \beta b_t \, d\mu. \tag{27}$$

Note that the family $q(a-\alpha b)-\mathcal{F}_u(\alpha)\bar{u}$ is almost additive and has tempered variation. Since

$$\sup_{t\in[0,s]} \|a_t\|_\infty < \infty, \quad \sup_{t\in[0,s]} \|b_t\|_\infty < \infty, \quad \sup_{t\in[0,s]} \|u_t\|_\infty < \infty$$

for some number $s > 0$, we also have

$$\sup_{t\in[0,s]} \|q(a_t - \alpha b_t) - \mathcal{F}_u(\alpha) u_t\|_\infty < \infty.$$

It follows from Theorem 5 that

$$\begin{aligned} L(q) &\geq h_\mu(\Phi) + \lim_{t\to\infty} \frac{1}{t} \int_X [q(a_t - \alpha b_t) - \mathcal{F}_u(\alpha) u_t] \, d\mu \\ &= h_\mu(\Phi) + \lim_{t\to\infty} \frac{1}{t} q \int_X (a_t - \alpha b_t) \, d\mu - \mathcal{F}_u(\alpha) \lim_{t\to\infty} \frac{1}{t} \int_X u_t \, d\mu. \end{aligned} \tag{28}$$

Since

$$q \int_X (\beta - \alpha) b_t \, d\mu = \frac{q(\operatorname{sgn} q) r}{2} \int_X b_t \, d\mu = \frac{|q| r}{2} \int_X b_t \, d\mu,$$

we have

$$\frac{1}{t} q \int_X (a_t - \alpha b_t) \, d\mu = \frac{1}{t} q \int_X (a_t - \beta b_t) \, d\mu + \frac{|q| r}{2} \frac{1}{t} \int_X b_t \, d\mu.$$

Finally, letting $t \to \infty$ and using (27), we obtain

$$\lim_{t\to\infty} \frac{1}{t} q \int_X (a_t - \alpha b_t) \, d\mu = \frac{|q| r}{2} \lim_{t\to\infty} \frac{1}{t} \int_X b_t \, d\mu. \tag{29}$$

By Birkhoff's ergodic theorem, we have

$$\lim_{t\to\infty} \frac{1}{t} \int_X u_t \, d\mu = \int_X \lim_{t\to\infty} \frac{1}{t} \int_0^t (u \circ \phi_s) \, ds \, d\mu = \int_X u \, d\mu$$

and since $h_\mu(\Phi) \geq 0$, it follows from (28) and (29) that

$$L(q) \geq \frac{|q| r}{2} \lim_{t\to\infty} \frac{1}{t} \int_X b_t \, d\mu - \mathcal{F}_u(\alpha) \int_X u \, d\mu. \tag{30}$$

On the other hand, by (14) we have

$$\lim_{t\to\infty} \frac{1}{t} \int_X b_t \, d\mu > 0.$$

This implies that the right-hand side of (30) takes arbitrary large values for $|q|$ large and so there exists $\bar{q} > 0$ with $L(q) \geq L(0)$ for $q \in \mathbb{R}$ with $|q| \geq \bar{q}$.

Since $\operatorname{span}\{a, b, \bar{u}\} \subset E(X)$, by Proposition 6 the function $q \mapsto L(q)$ is of class C^1. Therefore, L attains a minimum at some point $q = q(\alpha)$ with $|q(\alpha)| \leq \bar{q}$ and so $(dL/dq)(q(\alpha)) = 0$. Now let μ_α be the equilibrium measure of the family $q(\alpha)(a - \alpha b) - \mathcal{F}_u(\alpha)\bar{u}$. Again by Proposition 6, we have

$$0 = \frac{d}{dq} L(q)|_{q=q(\alpha)} = \lim_{t\to\infty} \frac{1}{t} \int_X (a_t - \alpha b_t)\, d\mu_\alpha \tag{31}$$

and thus,

$$\mathcal{P}(\mu_\alpha) = \frac{\lim_{t\to\infty} \frac{1}{t} \int_X a_t\, d\mu_\alpha}{\lim_{t\to\infty} \frac{1}{t} \int_X b_t\, d\mu_\alpha} = \alpha.$$

Moreover,

$$\begin{aligned} L(q(\alpha)) &= P\big(q(\alpha)(a - \alpha b) - \mathcal{F}_u(\alpha)\bar{u}\big) \\ &= h_{\mu_\alpha}(\Phi) + \lim_{t\to\infty} \frac{1}{t} \int_X (a_t - \alpha b_t)\, d\mu_\alpha - \mathcal{F}_u(\alpha) \lim_{t\to\infty} \frac{1}{t} \int_X u_t\, d\mu_\alpha \\ &= h_{\mu_\alpha}(\Phi) - \mathcal{F}_u(\alpha) \int_X u\, d\mu_\alpha. \end{aligned}$$

It follows from Lemma 10 that $L(q(\alpha)) \geq 0$ and so

$$\mathcal{F}_u(\alpha) \leq \frac{h_{\mu_\alpha}(\Phi)}{\int_X u\, d\mu_\alpha}. \tag{32}$$

Proposition 6 also says that μ_α is ergodic and so it follows from (5) that

$$\dim_u \mu_\alpha = \frac{h_{\mu_\alpha}(\Phi)}{\int_X u\, d\mu_\alpha}. \tag{33}$$

By (31) and Proposition 4, we have $\mu_\alpha(K_\alpha) = 1$. This implies that

$$\dim_u \mu_\alpha \leq \dim_u K_\alpha = \mathcal{F}_u(\alpha),$$

which together with (32) and (33) gives $\dim_u \mu_\alpha = \mathcal{F}_u(\alpha)$. Hence,

$$\begin{aligned} \min_{q\in\mathbb{R}} P\big(q(a - \alpha b) - \mathcal{F}_u(\alpha)\bar{u}\big) &= L(q(\alpha)) \\ &= h_{\mu_\alpha}(\Phi) - \mathcal{F}_u(\alpha) \int_X u\, d\mu_\alpha \\ &= h_{\mu_\alpha(\Phi)} - \frac{h_{\mu_\alpha(\Phi)}}{\int_X u\, d\mu_\alpha} \int_X u\, d\mu_\alpha = 0, \end{aligned} \tag{34}$$

which yields the desired statement. □

We proceed with the proof of the theorem. Take $\alpha \in \mathbb{R}$ such that $K_\alpha \neq \emptyset$. Given $x \in X$, we consider the family $(\mu_{x,t})_{t>0}$ of probability measures on X defined by

$$\mu_{x,t} = \frac{1}{t} \int_0^t \delta_{\phi_s(x)}\, ds,$$

where δ_y is the probability measure concentrated at y. Let $V(x)$ be the set of all sublimits of this family in the weak* topology. Then $\emptyset \neq V(x) \subset \mathcal{M}_\Phi$.

For each $\mu \in V(x)$ there exists an increasing sequence $(t_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} \frac{a_{t_n}(x)}{t_n} = \lim_{t\to\infty} \frac{1}{t} \int_X a_t \, d\mu \quad \text{and} \quad \lim_{n\to\infty} \frac{b_{t_n}(x)}{t_n} = \lim_{t\to\infty} \frac{1}{t} \int_X b_t \, d\mu \tag{35}$$

(see [9]). It follows from (24) and (35) that

$$\mathcal{P}(\mu) = \lim_{t\to\infty} \frac{\int_X a_t \, d\mu}{\int_X b_t \, d\mu} = \frac{\lim_{t\to\infty} \frac{1}{t} \int_X a_t \, d\mu}{\lim_{t\to\infty} \frac{1}{t} \int_X b_t \, d\mu} = \lim_{n\to\infty} \frac{a_{t_n}(x)}{b_{t_n}(x)} = \alpha,$$

which shows that $\alpha \in \mathcal{P}(\mathcal{M}_\Phi)$.

Now take $\alpha \in \operatorname{int} \mathcal{P}(\mathcal{M}_\Phi)$ and $\mu \in \mathcal{M}_\Phi$ such that $\mathcal{P}(\mu) = \alpha$. By Proposition 4, the map

$$\mu \mapsto \lim_{t\to\infty} \frac{1}{t} \int_X u_t \, d\mu = \int_X u \, d\mu$$

is continuous and by hypothesis, the map $\mu \mapsto h_\mu(\Phi)$ is upper semicontinuous. Therefore, the function $\mu \mapsto h_\mu(\Phi)/\int_X u \, d\mu$ is upper semicontinuous. Since $\mathcal{P}$ is continuous and $\mathcal{M}_\Phi$ is compact, there exists the maximum in (22). By Theorem 5 and Lemma 11, we have

$$\begin{aligned} 0 &= \inf_{q\in\mathbb{R}} P\big(q(a-\alpha b) - \mathcal{F}_u(\alpha)\bar{u}\big) \\ &\geq \inf_{q\in\mathbb{R}} \left\{ h_\mu(\Phi) + \lim_{t\to\infty} \frac{1}{t} \int_X [q(a_t - \alpha b_t) - \mathcal{F}_u(\alpha) u_t] \, d\mu \right\}. \end{aligned}$$

It follows from $\mathcal{P}(\mu) = \alpha$ that

$$\lim_{t\to\infty} \frac{1}{t} \int_X (a_t - \alpha b_t) \, d\mu = 0$$

and so

$$\begin{aligned} 0 &\geq \inf_{q\in\mathbb{R}} \left\{ h_\mu(\Phi) - \lim_{t\to\infty} \frac{1}{t} \int_X \mathcal{F}_u(\alpha) u_t \, d\mu \right\} \\ &= h_\mu(\Phi) - \mathcal{F}_u(\alpha) \lim_{t\to\infty} \frac{1}{t} \int_X u_t \, d\mu \\ &= h_\mu(\Phi) - \mathcal{F}_u(\alpha) \int_X u \, d\mu. \end{aligned}$$

Hence,

$$\mathcal{F}_u(\alpha) \geq \frac{h_\mu(\Phi)}{\int_X u \, d\mu}. \tag{36}$$

Again by Lemma 11, there exists an ergodic equilibrium measure μ_α such that $\mathcal{P}(\mu_\alpha) = \alpha$, $\mu_\alpha(K_\alpha) = 1$ and

$$\mathcal{F}_u(\alpha) = \frac{h_{\mu_\alpha}(\Phi)}{\int_X u \, d\mu_\alpha} = \dim_u \mu_\alpha.$$

Together with (36) this yields identities (22) and (23). In particular, $K_\alpha \neq \emptyset$.

Now we establish property 2 in the theorem. By (34), we have

$$P\big(q(\alpha)(a-\alpha b) - \mathcal{F}_u(\alpha)\bar{u}\big) = \min_{q\in\mathbb{R}} P\big(q(a-\alpha b) - \mathcal{F}_u(\alpha)\bar{u}\big) = 0,$$

which gives

$$\mathcal{F}_u(\alpha) = S_u(\alpha, q(\alpha)) \geq \inf_{q\in\mathbb{R}} \big\{ S_u(\alpha, q) : q \in \mathbb{R} \big\}.$$

On the other hand, by the definition of S_u and again by Lemma 11, we obtain

$$P\big(q(a-\alpha b)-S_u(\alpha,q)\bar{u}\big)=0\le P\big(q(a-\alpha b)-\mathcal{F}_u(\alpha)\bar{u}\big)$$

for every $q\in\mathbb{R}$. Let μ_q be the equilibrium measure of $q(a-\alpha b)-\mathcal{F}_u(\alpha)\bar{u}$. Then

$$\begin{aligned}
0&\le P\big(q(a-\alpha b)-\mathcal{F}_u(\alpha)\bar{u}\big)-P\big(q(a-\alpha b)-S_u(\alpha,q)\bar{u}\big)\\
&\le P\big(q(a-\alpha b)-\mathcal{F}_u(\alpha)\bar{u}\big)\\
&\quad-\left[h_{\mu_q}(\Phi)+\lim_{t\to\infty}\frac{1}{t}\int_X[q(a_t-\alpha b_t)-S_u(\alpha,q)u_t]\,d\mu_q\right]\\
&=h_{\mu_q}(\Phi)+\lim_{t\to\infty}\frac{1}{t}\int_X[q(a_t-\alpha b_t)-\mathcal{F}_u(\alpha)u_t]\,d\mu_q-h_{\mu_q}(\Phi)\\
&\quad-\lim_{t\to\infty}\frac{1}{t}\int_X[q(a_t-\alpha b_t)]\,d\mu_q+S_u(\alpha,q)\lim_{t\to\infty}\frac{1}{t}\int_X u_t\,d\mu_q\\
&=\big[S_u(\alpha,q)-\mathcal{F}_u(\alpha)\big]\lim_{t\to\infty}\frac{1}{t}\int_X u_t\,d\mu_q\\
&=\big[S_u(\alpha,q)-\mathcal{F}_u(\alpha)\big]\int_X u\,d\mu_q.
\end{aligned}$$

Since $u>0$, we obtain $S_u(\alpha,q)-\mathcal{F}_u(\alpha)\ge 0$ for every $q\in\mathbb{R}$. Therefore,

$$\mathcal{F}_u(\alpha)\le\inf\big\{S_u(\alpha,q):q\in\mathbb{R}\big\}.$$

Finally, we show that the spectrum is continuous. Given $\alpha\in\operatorname{int}\mathcal{P}(\mathcal{M}_\Phi)$, let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence in $\operatorname{int}\mathcal{P}(\mathcal{M}_\Phi)$ converging to α. Given $n\in\mathbb{N}$, take $q_n\in\mathbb{R}$ such that $\mathcal{F}_u(\alpha_n)=S_u(\alpha_n,q_n)$ and take $q(\alpha)\in\mathbb{R}$ such that $\mathcal{F}_u(\alpha)=S_u(\alpha,q(\alpha))$. By the second property in the theorem, we have

$$\mathcal{F}_u(\alpha_n)=\min_{q\in\mathbb{R}}S_u(\alpha_n,q)\le S_u(\alpha_n,q(\alpha)).\tag{37}$$

On the other hand, by Proposition 6, the function

$$(q,\alpha,p)\mapsto P\big(q(a-\alpha b)-p\bar{u}\big)$$

is of class C^1 and by the Implicit function theorem, $(\alpha,q)\mapsto S_u(\alpha,q)$ is also of class C^1. It follows from (37) that

$$\limsup_{n\to\infty}\mathcal{F}_u(\alpha_n)\le\limsup_{n\to\infty}S_u(\alpha_n,q(\alpha))=S_u(\alpha,q(\alpha))=\mathcal{F}_u(\alpha).$$

Moreover, since $\mathcal{F}_u(\alpha)$ is a minimum, we have

$$\mathcal{F}_u(\alpha)=\min_{q\in\mathbb{R}}S_u(\alpha,q)\le S_u(\alpha,q_n)$$

for every $n\in\mathbb{N}$ and so

$$\mathcal{F}_u(\alpha)\le\liminf_{n\to\infty}S_u(\alpha_n,q_n)=\liminf_{n\to\infty}\mathcal{F}_u(\alpha_n).$$

This completes the proof of the theorem. □

We are not aware whether the ergodic measures μ_α satisfying (23) that are constructed in the proof are unique under reasonable general assumptions. Any such assumptions on this respect would be quite welcome.

5. Applications of Theorem 9

In this section we describe a few classes of flows and of almost additive families of continuous functions to which Theorem 9 applies. We start by introducing the notions of hyperbolicity and Markov system.

5.1. Hyperbolic sets and Markov systems.

Let Φ be a C^1 flow on a smooth manifold M. A compact Φ-invariant set $\Lambda \subset M$ is called a *hyperbolic set for* Φ if there exists a splitting

$$T_\Lambda M = E^s \oplus E^u \oplus E^0$$

and constants $c > 0$ and $\lambda \in (0,1)$ such that for each $x \in \Lambda$:

(1) the vector $(d/dt)\phi_t(x)|_{t=0}$ generates $E^0(x)$;
(2) for each $t \in \mathbb{R}$ we have
$$d_x\phi_t E^s(x) = E^s(\phi_t(x)) \quad \text{and} \quad d_x\phi_t E^u(x) = E^u(\phi_t(x));$$
(3) $\|d_x\phi_t v\| \le c\lambda^t \|v\|$ for $v \in E^s(x)$ and $t > 0$;
(4) $\|d_x\phi_{-t} v\| \le c\lambda^t \|v\|$ for $v \in E^u(x)$ and $t > 0$.

Given a hyperbolic set Λ and $\varepsilon > 0$, for each $x \in \Lambda$ let $V^s(x)$ and $V^u(x)$ be the largest connected components of the sets

$$A^s(x) = \{y \in B(x,\varepsilon) : d(\phi_t(y), \phi_t(x)) \searrow 0 \text{ when } t \to +\infty\}$$

and

$$A^u(x) = \{y \in B(x,\varepsilon) : d(\phi_t(y), \phi_t(x)) \searrow 0 \text{ when } t \to -\infty\}$$

that contain x. Given a locally maximal hyperbolic set Λ (this means that $\Lambda = \bigcap_{t\in\mathbb{R}} \phi_t(U)$ for some open neighborhood U of Λ) and a sufficiently small $\varepsilon > 0$, there exists $\delta > 0$ such that if $x, y \in \Lambda$ satisfy $d(x,y) \le \delta$, then there exists a unique $t = t(x,y) \in [-\varepsilon, \varepsilon]$ such that

$$[x,y] := V^s(\phi_t(x)) \cap V^u(y)$$

is a single point in Λ.

Now we recall the notion of a Markov system. Consider an open smooth disk $D \subset M$ of dimension $\dim M - 1$ that is transverse to Φ and take $x \in D$. Let $U(x)$ be an open neighborhood of x diffeomorphic to $D \times (-\varepsilon, \varepsilon)$. We say that a closed set $R \subset \Lambda \cap D$ is a *rectangle* if $R = \overline{\operatorname{int} R}$ and $\pi_D([x,y]) \in R$ for $x, y \in R$. Now consider rectangles $R_1, \ldots, R_k \subset \Lambda$ such that

$$R_i \cap R_j = \partial R_i \cap \partial R_j \quad \text{for } i \ne j$$

and let $Z = \bigcup_{i=1}^k R_i$. We assume that $\Lambda = \bigcup_{t\in[0,\varepsilon]} \phi_t(Z)$ and that either

$$\phi_t(R_i) \cap R_j = \emptyset \quad \text{for all } t \in [0,\varepsilon]$$

or

$$\phi_t(R_j) \cap R_i = \emptyset \quad \text{for all } t \in [0,\varepsilon]$$

whenever $i \ne j$. We define a function $\tau \colon \Lambda \to \mathbb{R}_0^+$ by

$$\tau(x) = \min\{t > 0 : \phi_t(x) \in Z\}$$

and a map $T\colon \Lambda \to Z$ by $T(x) = \phi_{\tau(x)}(x)$. The restriction T_Z of T to Z is invertible and $T^n(x) = \phi_{\tau_n(x)}(x)$, where

$$\tau_n(x) = \sum_{i=0}^{n-1} \tau(T^i(x)).$$

The collection $R_1, \ldots, R_k$ is said to be a *Markov system* for Φ on Λ if

$$T(\operatorname{int}(V^s(x) \cap R_i)) \subset \operatorname{int}(V^s(T(x)) \cap R_j)$$

and

$$T^{-1}(\operatorname{int}(V^u(T(x)) \cap R_j)) \subset \operatorname{int}(V^u(x) \cap R_i)$$

for every $x \in \operatorname{int} T(R_i) \cap \operatorname{int} R_j$ and $i, j = 1, \ldots, k$. By work of Bowen [14] and Ratner [21], any locally maximal hyperbolic set Λ has Markov systems of arbitrarily small diameter and the function τ is always Hölder continuous on each domain of continuity.

Given a Markov system $R_1, \ldots, R_k$ for a flow Φ on a locally maximal hyperbolic set Λ, we consider the $k \times k$ matrix A with entries

$$a_{ij} = \begin{cases} 1 & \text{if } \operatorname{int} T(R_i) \cap R_j \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

We also consider the set

$$\Sigma_A = \bigl\{ (\cdots i_{-1} i_0 i_1 \cdots) : a_{i_n i_{n+1}} = 1 \text{ for } n \in \mathbb{Z} \bigr\} \subset \{1, \ldots, k\}^{\mathbb{Z}}$$

and the shift map $\sigma\colon \Sigma_A \to \Sigma_A$ defined by $\sigma(\cdots i_0 \cdots) = (\cdots j_0 \cdots)$, where $j_n = i_{n+1}$ for each $n \in \mathbb{Z}$. Finally, we define a *coding map* $\pi\colon \Sigma_A \to Z$ by

$$\pi(\cdots i_0 \cdots) = \bigcap_{n \in \mathbb{Z}} R_{i_{-n} \cdots i_n},$$

where $R_{i_{-n} \cdots i_n} = \bigcap_{l=-n}^{n} \overline{T_Z^{-l} \operatorname{int} R_{i_l}}$. The following properties hold:

(1) $\pi \circ \sigma = T \circ \pi$;
(2) π is Hölder continuous and onto;
(3) π is one-to-one on a full measure set with respect to any ergodic measure of full support and on a residual set.

Now let μ be a T_Z-invariant probability measure on Z. One can show that μ induces a Φ-invariant probability measure ν on Λ such that

$$\int_\Lambda g \, d\nu = \frac{\int_Z \int_0^{\tau(x)} (g \circ \phi_s)(x) \, ds \, d\mu}{\int_Z \tau \, d\mu} \tag{38}$$

for any continuous function $g\colon \Lambda \to \mathbb{R}$. Moreover, any Φ-invariant probability measure ν on Λ is of this form for some T_Z-invariant probability measure μ on Z. Abramov's entropy formula says that

$$h_\nu(\Phi) = \frac{h_\mu(T_Z)}{\int_Z \tau \, d\mu}. \tag{39}$$

By (38) and (39) we obtain

$$h_\nu(\Phi) + \int_\Lambda g \, d\nu = \frac{h_\mu(T_Z) + \int_Z I_g \, d\mu}{\int_Z \tau \, d\mu}, \tag{40}$$

where $I_g(x) = \int_0^{\tau(x)} (g \circ \phi_s)(x) \, ds$.

5.2. **Examples.** In this section we describe two scenarios to which Theorem 9 applies: locally maximal hyperbolic sets and suspension flows over expansive maps with specification.

For the second scenario, we need to recall the notion of a suspension flow. Let $T\colon X \to X$ be a homeomorphism on a compact metric space and let $\tau\colon X \to \mathbb{R}^+$ be a Lipschitz function. Consider the space

$$W = \{(x, s) \in X \times \mathbb{R} : 0 \le s \le \tau(x)\}$$

and let Y be the set obtained from W identifying $(x, \tau(x))$ and $(T(x), 0)$ for each $x \in X$. Then a certain distance introduced by Bowen and Walters in [16] makes Y a compact topological space. The *suspension flow* over T with height function τ is the flow $\Psi = (\psi)_{t\in\mathbb{R}}$ on Y with the maps $\psi_t : Y \to Y$ defined by $\psi_t(x, s) = (x, s + t)$. We note that the identities in (38), (39) and (40) still hold for suspension flows (with Λ and Z replaced, respectively, by the sets Y and X).

Now we present two scenarios to which Theorem 9 applies.

Locally maximal hyperbolic sets. If Λ is a locally maximal hyperbolic set for a C^1 flow Φ such that $\Phi|_\Lambda$ is topologically mixing, then the entropy map $\mu \mapsto h_\mu(\Phi|_\Lambda)$ is upper-semicontinuous. Moreover, if the almost additive families of continuous functions a and b have bounded variation and u is Hölder continuous, then Theorem 12 in [9] shows that $\text{span}\{a, b, \overline{u}\} \subset E(\Lambda)$. We recall that a family of functions a is said to have *bounded variation* if for every $\kappa > 0$ there exists $\varepsilon > 0$ such that

$$|a_t(x) - a_t(y)| < \kappa \quad \text{whenever } y \in B_t(x, \varepsilon).$$

Suspension flows over expansive maps with specification. Let $X = (X, d)$ be a compact metric space and let $T\colon X \to X$ be a continuous expansive map with the specification property. We recall that a map T is said to be *expansive* if there exists $c > 0$ such that if $d(T^n(x), T^n(y)) < c$ for all $n \ge 0$, then $x = y$. Moreover, a map T is said to have the *specification property* if for each $\varepsilon > 0$ there exists $m = m(\varepsilon) \in \mathbb{N}$ such that given intervals $I_j = [a_j, b_j]$ with $a_j, b_j \in \mathbb{N}$ for $j = 1, \ldots, k$ such that

$$d(I_i, I_j) \ge m(\varepsilon) \quad \text{whenever } i \ne j$$

and given points $x_1, \ldots, x_k \in X$ there exists $x \in X$ such that

$$d(T^{p+a_j}(x), T^p(x_j)) < \varepsilon$$

for $p = 0, \ldots, b_j - a_j$ and $j = 1, \ldots, k$. Examples of continuous expansive maps T with the specification property include for example repellers, locally maximal hyperbolic sets and topological Markov chains.

Now let Φ be a suspension flow over such a map T. Then Φ is an expansive flow and so the entropy map $\mu \mapsto h_\mu(\Phi)$ is upper-semicontinuous. Moreover, if the almost additive families of continuous functions a and b as well as $\overline{u}$ have bounded variation, then one can show that $\text{span}\{a, b, \overline{u}\} \subset E(X)$.

5.3. **Cocycles and top Lyapunov exponent.** Let $\Phi = (\phi_t)_{t\in\mathbb{R}}$ be a continuous flow on a compact metric space M. Moreover, let $GL(d, \mathbb{R})$ be the set of all invertible $d \times d$ matrices. A continuous map $A\colon \mathbb{R} \times M \to GL(d, \mathbb{R})$ is called a *linear cocycle* over Φ if for all $t, s \in \mathbb{R}$ and $x \in X$ we have:

(1) $A(0,x) = \mathrm{Id}$;
(2) $A(t+s,x) = A(s,\phi_t(x))A(t,x)$.

We shall always assume that all entries $a_{ij}(t,x)$ of $A(t,x)$ are positive for every $(t,x) \in \mathbb{R} \times M$. Moreover, for definiteness we shall consider the norm on $GL(d,\mathbb{R})$ defined by $\|A\| = \sum_{i,j=1}^d |a_{ij}|$, denoting by a_{ij} the entries of A.

Now we consider the family of functions $a := (a_t)_{t\geq 0}$ defined by

$$a_t(x) = \log \|A(t,x)\|.$$

Proposition 12. *The sequence a is almost additive with respect to Φ.*

Proof. To the possible extent, we follow the proof of Lemma 2.1 in [17]. Take $s_0 > 0$. Since the map $(t,x) \mapsto A(t,x)$ is continuous, we have

$$\sup_{(t,x)\in N} \|A(t,x)\| < \infty,$$

where $N = [0,s_0] \times M$. Together with the assumption that all entries of the matrices $A(t,x)$ are positive, this implies that

$$\min_{(t,x)\in N} a_{ij}(t,x) > 0 \quad \text{and} \quad \max_{(t,x)\in N} a_{ij}(t,x) < \infty$$

for all i,j. Hence, there exists $K > 0$ (possibly depending on s_0) such that

$$1 \geq \frac{\min_{(t,x)\in N} a_{ij}(t,x)}{\max_{(t,x)\in N} a_{kl}(t,x)} \geq K$$

for all i,j,k,l. Then

$$a_{ij}(t,x) \geq c(JA(t,x))_{ij} \quad \text{for each } i,j = 1,\dots,d,$$

where $c = K/d$ and where J is the $d \times d$ matrix with all entries equal 1. Now note that

$$A(t,x) = A(t-s_0+s_0,x) = A(s_0,x)A(t-s_0,\phi_{s_0}(x)).$$

Hence, denoting by z the $d \times 1$ vector with all entries equal to 1, we have

$$\begin{aligned}
\|A(t+s,x)\| &= \|A(s,\phi_t(x))A(s_0,x)A(t-s_0,\phi_{s_0}(x))\| \\
&\geq \|A(s,\phi_t(x))cJA(s_0,x)A(t-s_0,\phi_{s_0}(x))\| \\
&= c\|A(s,\phi_t(x))JA(t,x)\| \\
&= cz^tA(s,\phi_t(x))JA(t,x)z \\
&= c(z^tA(s,\phi_t(x))z)(z^tA(t,x)z) \\
&= c\|A(s,\phi_t(x))\| \cdot \|A(t,x)\|.
\end{aligned}$$

Then

$$\log\|A(t+s,x)\| \geq \log c + \log\|A(s,\phi_t(x))\| + \log\|A(t,x)\|$$

and since $K \leq 1$, we obtain $C = -\log c > 0$. On the other hand, we have

$$\|A(t+s,x)\| \leq \|A(s,\phi_t(x))\| \cdot \|A(t,x)\|$$

for every $t,s \in \mathbb{R}$ and $x \in M$. Therefore,

$$-C + a_t(x) + a_s(\phi_t(x)) \leq a_{t+s}(x) \leq a_t(x) + a_s(\phi_t(x)) + C$$

for every $t,s \geq 0$ and $x \in M$. □

Following [7], we say that A has *tempered distortion* if

$$\limsup_{t\to\infty}\frac{1}{t}\log\sup\bigl\{\|A(t,x)A(t,y)^{-1}\| : z\in M \text{ and } x,y\in B_t(z,\varepsilon)\bigr\}=0$$

for some $\varepsilon>0$. Moreover, we say that A has *bounded distortion* if

$$\sup\bigl\{\|A(t,x)A(t,y)^{-1}\| : z\in M \text{ and } x,y\in B_t(z,\varepsilon)\bigr\}<\infty$$

for some $\varepsilon>0$. Clearly, bounded distortion implies tempered distortion.

Now observe that

$$\|A(t,x)A(t,x)^{-1}\|=\|\mathrm{Id}\|=d$$

for every $(t,x)\in\mathbb{R}\times M$, which implies that

$$\|A(t,x)^{-1}\|\ge d\|A(t,x)\|^{-1}.$$

Therefore,

$$\begin{aligned}\|A(t,x)A(t,y)^{-1}\| &\ge \frac{K}{d}\|A(t,x)\|\cdot\|A(t,y)^{-1}\|\\ &\ge K\|A(t,x)\|\cdot\|A(t,y)\|^{-1}\end{aligned}$$

and so

$$\bigl|\log\|A(t,x)\|-\log\|A(t,y)\|\bigr|\le -\log K+\log\|A(t,x)A(t,y)^{-1}\|.$$

In particular, for $z\in M$ and $\varepsilon>0$ we have

$$\sup_{x,y\in B_t(z,\varepsilon)}|a_t(x)-a_t(y)|\le -\log K+\log\sup_{x,y\in B_t(z,\varepsilon)}\|A(t,x)A(t,y)^{-1}\|.$$

Hence, if A has tempered distortion, then a has tempered variation, and if A has bounded distortion, then a has bounded variation.

For a specific example, one can consider a C^1 flow Φ on a compact set $M\subset\mathbb{R}^d$ such that for every $t\in\mathbb{R}$ and $x\in M$ the matrix $d_x\phi_t$ has only positive entries. Then $A(t,x)=d_x\phi_t$ is a linear cocycle over Φ and the family a defined by $a_t(x)=\log\|d_x\phi_t\|$ is an almost additive family of continuous functions with respect to Φ (by Proposition 12). In particular, for the family b defined by $b_t=t$, the set K_α in (15) is a level set of the top Lyapunov exponent for the flow Φ. Thus, in this case Theorem 9 can be applied to give a multifractal analysis of the top Lyapunov exponent.

6. Irregular sets

In this section we study the u-dimension of the irregular sets obtained from almost additive families of functions on locally maximal hyperbolic sets.

Let $\Lambda\subset M$ be a locally maximal hyperbolic set for a C^1 flow Φ such that $\Phi|_\Lambda$ is topologically mixing and let u be a Hölder continuous function with $P_\Phi(u)=0$. Moreover, we consider two families of continuous functions $a,b\in A(M)$.

Given a Markov system $R_1,\dots,R_k$ for Φ on Λ, let $Z=\bigcup_{i=1}^k R_i$ and consider the associated map T_Z introduced in Section 5.1. We also consider the sequences of functions $c=(c_n)_{n\in\mathbb{N}}$ and $d=(d_n)_{n\in\mathbb{N}}$ on Z defined by

$$c_n(x)=a_{\tau_n(x)}(x)\quad\text{and}\quad d_n(x)=b_{\tau_n(x)}(x)$$

for every $x\in Z$ and $n\in\mathbb{N}$. The families of functions c and d are almost additive with respect to the map T_Z (see Lemma 9 in [9]).

We also consider the *irregular sets* given by

$$C_\Phi = \left\{ x \in \Lambda : \liminf_{t\to\infty} \frac{a_t(x)}{b_t(x)} < \limsup_{t\to\infty} \frac{a_t(x)}{b_t(x)} \right\}$$

and

$$C_{T_Z} = \left\{ x \in Z : \liminf_{n\to\infty} \frac{c_n(x)}{d_n(x)} < \limsup_{n\to\infty} \frac{c_n(x)}{d_n(x)} \right\}.$$

Since the coding map $\pi : \Sigma_A \to Z$ is onto, we have $\pi(C_\sigma) = C_{T_Z}$, where

$$C_\sigma = \left\{ \omega \in \Sigma_A : \liminf_{n\to\infty} \frac{(c_n \circ \pi)(\omega)}{(d_n \circ \pi)(\omega)} < \limsup_{n\to\infty} \frac{(c_n \circ \pi)(\omega)}{(d_n \circ \pi)(\omega)} \right\}.$$

The following theorem is the main result of this section.

Theorem 13. *Let $\Lambda \subset M$ be a locally maximal hyperbolic set for a C^1 flow Φ such that $\Phi|_\Lambda$ is topologically mixing and let u be a Hölder continuous function with $P_\Phi(u) = 0$. If $a, b, \bar{u} \in A(M)$ are families of continuous functions with bounded variation such that $\mathcal{P}(\nu_u) \in \operatorname{int} \mathcal{P}(\mathcal{M}_\Phi)$ for the unique equilibrium measure ν_u for u, then*

$$\dim_u C_\Phi = \dim_u \Lambda.$$

Proof. By (40), if $P_\Phi(u) = 0$, then ν_u is the unique equilibrium measure for u if and only if the induced measure μ_u on Z is the unique equilibrium measure for I_u. Since every linear combination v of the families a, b and $\bar{u}$ has bounded variation and satisfies $\sup_{t\in[0,s]} \|v_t\|_\infty < \infty$ for some $s > 0$, it follows from Theorem 12 in [9] that $\operatorname{span}\{a, b, \bar{u}\} \subset E(M)$.

We continue with an auxiliary result.

Lemma 14. *For every $x \in Z$ we have*

$$\limsup_{t\to\infty} \frac{a_t(x)}{b_t(x)} = \limsup_{n\to\infty} \frac{c_n(x)}{d_n(x)} \tag{41}$$

and

$$\liminf_{t\to\infty} \frac{a_t(x)}{b_t(x)} = \liminf_{n\to\infty} \frac{c_n(x)}{d_n(x)}. \tag{42}$$

Proof of the lemma. For each $t > 0$ there exists a unique $n \in \mathbb{N}$ such that $\tau_n(x) \le t < \tau_{n+1}(x)$ and so $t - \tau_n(x) \in [0, \sup \tau)$. Since the families a and b are almost additive, there exists $C > 0$ such that

$$-C + a_{\tau_n(x)}(x) + a_{t-\tau_n(x)}(x) \le a_t(x) \le a_{\tau_n(x)}(x) + a_{t-\tau_n(x)}(x) + C$$

and

$$-C + b_{\tau_n(x)}(x) + b_{t-\tau_n(x)}(x) \le b_t(x) \le b_{\tau_n(x)}(x) + b_{t-\tau_n(x)}(x) + C.$$

This implies that

$$\left| a_t(x) - a_{\tau_n(x)}(x) \right| \le \sup_{s\in[0,\sup\tau]} \|a_s\|_\infty + C =: q_1 < \infty \tag{43}$$

and

$$\left| b_t(x) - b_{\tau_n(x)}(x) \right| \le \sup_{s\in[0,\sup\tau]} \|b_s\|_\infty + C =: q_2 < \infty. \tag{44}$$

It follows from (14), (43) and (44) that

$$\frac{a_t(x)}{b_t(x)} \le \frac{a_{\tau_n(x)}(x)}{b_{\tau_n(x)}(x) - q_2} + \frac{q_1}{b_{\tau_n(x)}(x) - q_2}$$

and

$$\frac{a_t(x)}{b_t(x)} \ge \frac{a_{\tau_n(x)}(x)}{b_{\tau_n(x)}(x) + q_2} - \frac{q_1}{b_{\tau_n(x)}(x) + q_2}$$

for all sufficiently large n. Finally, since $t \to \infty$ implies $n \to \infty$ ($\tau_n(x) \to \infty$) and vice-versa, we conclude that (41) and (42) hold for every $x \in Z$. □

It follows directly from Proposition 8 that the set C_Φ is Φ-invariant. By Lemma 14 and again by Proposition 8, we obtain

$$\limsup_{n\to\infty} \frac{c_n(T_Z(x))}{d_n(T_Z(x))} = \limsup_{n\to\infty} \frac{c_n(x)}{d_n(x)}$$

and

$$\liminf_{n\to\infty} \frac{c_n(T_Z(x))}{d_n(T_Z(x))} = \liminf_{n\to\infty} \frac{c_n(x)}{d_n(x)},$$

which implies that the set C_{T_Z} is T_Z-invariant.

We also recall the notion of u-dimension for maps. Let $f\colon X \to X$ be a continuous map and let $\mathcal{V}$ be a finite open cover of X. Given $k \in \mathbb{N}$, we denote by $\mathcal{W}_k(\mathcal{V})$ the collection of strings $(V_1, V_2, \ldots, V_k)$ of elements of $\mathcal{V}$. For each $V \in \mathcal{W}_k(\mathcal{V})$, we consider the open set

$$X(V) = \bigl\{x \in X : x \in V_1, f(x) \in V_2, \ldots, f^{k-1}(x) \in V_k\bigr\}.$$

Let $u\colon X \to \mathbb{R}$ be a positive continuous function. Given $B \subset X$ and $\alpha \in \mathbb{R}$, we define

$$N(B, \alpha, u, \mathcal{V}) = \lim_{n\to\infty} \inf_{\Gamma} \sum_{V\in\Gamma} \exp(-\alpha u(V)),$$

where

$$u(V) = \sup_{x\in X(V)} \sum_{k=0}^{n-1} u(f^k(x))$$

and with the infimum taken over all collections $\Gamma \subset \bigcup_{k\ge n} \mathcal{W}_k(\mathcal{V})$ such that $B \subset \bigcup_{V\in\Gamma} X(V)$. We also define

$$\dim_{u,\mathcal{V}} B = \inf\bigl\{\alpha \in \mathbb{R} : N(B, \alpha, u, \mathcal{V}) = 0\bigr\}.$$

One can show that the limit

$$\dim_u B = \lim_{\operatorname{diam} \mathcal{V}\to 0} \dim_{u,\mathcal{V}} B$$

exists and we call it the *u-dimension* of the set B with respect to f. Analogously, for $X = Z$ and a subset $B \subset Z$, we define

$$N_\alpha(B) = \lim_{l\to\infty} \inf_{\Gamma'} \sum_{R\in\Gamma'} \exp\left(-\alpha \sup_{x\in R} \sum_{k=0}^{m(R)-1} I_u(T_Z^k(x))\right),$$

where each Γ' is a cover of B by sets $R_{i_0\cdots i_m} = \bigcap_{l=0}^m \overline{T_Z^{-l} \operatorname{int} R_{i_l}}$ with $m \ge l$. Given $B \subset Z$, let

$$S_B = \bigl\{y = \phi_s(x) \in \Lambda : x \in B \text{ and } s \in [0, \tau(x)]\bigr\}.$$

Lemma 15. *For any set $B \subset Z$ we have*

$$\dim_u S_B = \inf\{\alpha \in \mathbb{R} : N_\alpha(B) = 0\}.$$

Proof of the lemma. For every $x \in \Lambda$ and $n \in \mathbb{N}$, we have

$$\begin{aligned}\int_0^{\tau_n(x)} (u \circ \phi_s)(x)ds &= \sum_{k=0}^{n-1} \int_{\tau_k(x)}^{\tau_{k+1}(x)} (u \circ \phi_s)(x)ds \\ &= \sum_{k=0}^{n-1} \int_0^{\tau(T^k(x))} (u \circ \phi_s)(T^k(x))ds \\ &= \sum_{k=0}^{n-1} (I_u \circ T^k)(x).\end{aligned} \tag{45}$$

Given $t > 0$, there exists a unique $n \in \mathbb{N}$ such that $t = \tau_n(x) + \kappa$ for some $\kappa \in [0, \sup \tau)$. Hence, it follows from (45) that

$$\begin{aligned}&\left| \int_0^t (u \circ \phi_s)(x)ds - \sum_{k=0}^{n-1} (I_u \circ T_Z^k)(x) \right| \\ &= \left| \int_0^t (u \circ \phi_s)(x)ds - \int_0^{\tau_n(x)} (u \circ \phi_s)(x)ds \right| \\ &= \left| \int_{\tau_n(x)}^t (u \circ \phi_s)(x)ds \right| \le (\sup u)(\sup \tau) =: \kappa.\end{aligned} \tag{46}$$

Assume that

$$\bigcup_{(x,t)\in\Gamma} B_t(x, \varepsilon) \supset S_B.$$

For each $(x, t) \in \Gamma$ take $\omega = (\cdots i_0(x) \cdots) \in \Sigma_A$ with $\pi(\omega) = x$. Moreover, let $R(x) = R_{i_0(x)\cdots i_{m(x)}(x)}$, where $m(x) \in \mathbb{N}$ is the unique integer such that $t = \tau_{m(x)}(x) + \kappa$ for some $\kappa \in [0, \sup \tau)$. Then

$$\Gamma' = \{R(x) : (x, t) \in \Gamma\}$$

is a cover of B and so it follows readily from (46) that

$$N_\alpha(B) \le e^\kappa N(S_B, u, \alpha, \varepsilon). \tag{47}$$

On the other hand, if the sets $R(x) = R_{i_0(x)\cdots i_{m(x)}(x)}$ form a cover of B for $x \in C \subset Z$ and some integers $m(x) \in \mathbb{N}$ for each $x \in C$, then

$$\bigcup_{x\in C} B_{\tau_{m(x)-1}(x)}(x, \varepsilon) \supset S_B$$

assuming that the elements of the Markov system have diameter at most ε. Again, it follows from (46) that

$$N(S_B, u, \alpha, \varepsilon) \le e^\kappa N_\alpha(B),$$

which together with (47) yields the desired result. □

Now we consider the map $\mathcal{P}_{c,d} \colon \mathcal{M}_{T_Z} \to \mathbb{R}$ given by

$$\mathcal{P}_{c,d}(\nu) = \frac{\lim_{n\to\infty} \frac{1}{n} \int_Z c_n \, d\nu}{\lim_{n\to\infty} \frac{1}{n} \int_Z d_n \, d\nu} = \frac{\lim_{n\to\infty} \int_Z c_n \, d\nu}{\lim_{n\to\infty} \int_Z d_n \, d\nu}.$$

Lemma 16. *Assume that T_Z is topologically mixing and let μ_u be the unique equilibrium measure of I_u. If $\mathcal{P}_{c,d}(\mu_u) \in \operatorname{int} \mathcal{P}_{c,d}(\mathcal{M}_{T_Z})$, then*

$$\dim_{I_u} C_{T_Z} = \dim_{I_u} Z.$$

Proof of the lemma. The map π is an homeomorphism on a set $\widetilde{Z} \subset Z$ satisfying $\mu_u(\widetilde{Z}) = 1$. Consider the measure m_u on Σ_A defined by $m_u(B) = \mu_u(\pi(B))$ for every Borel set $B \subset \Sigma_A$. Since μ_u is an equilibrium measure for I_u, we have

$$\begin{aligned} 0 = P_{T_Z}(I_u) &= h_{\mu_u}(T_Z) + \int_Z I_u \, d\mu_u \\ &= h_{\mu_u}(T_{\widetilde{Z}}) + \int_{\widetilde{Z}} I_u \, d\mu_u \\ &= h_{m_u}(\sigma) + \int_{\Sigma_A} I_u \circ \pi \, dm_u. \end{aligned}$$

Moreover,

$$0 = P_{T_Z}(I_u) \ge P_{T_{\widetilde{Z}}}(I_u) = P_\sigma(I_u \circ \pi).$$

This implies that

$$P_\sigma(I_u \circ \pi) = h_{m_u}(\sigma) + \int_{\Sigma_A} I_u \circ \pi \, dm_u = 0$$

and so m_u is an equilibrium measure for $I_u \circ \pi$. By Proposition 18 in [10], the function $I_u \circ \pi$ is Hölder continuous and since $\sigma|_{\Sigma_A}$ is topologically mixing, m_u is the unique equilibrium measure for $I_u \circ \pi$. Since $c \circ \pi$, $d \circ \pi$ and $\overline{I_u \circ \pi}$ have bounded variation, we also have

$$\operatorname{span}\{c \circ \pi, d \circ \pi, \overline{I_u \circ \pi}\} \subset E_\sigma(\Sigma_A).$$

Now consider the set

$$\pi^* \mathcal{M}_{T_Z} := \{m \in \mathcal{M}_\sigma : m = \pi^* \mu \text{ for some } \mu \in \mathcal{M}_{T_Z}\}.$$

Since $m_u \in \pi^* \mathcal{M}_{T_Z}$, we obtain

$$\mathcal{P}_{c\circ\pi, d\circ\pi}(m_u) = \mathcal{P}_{c,d}(\mu_u) \in \operatorname{int} \mathcal{P}_{c,d}(\mathcal{M}_{T_Z}) = \operatorname{int} \mathcal{P}_{c\circ\pi, d\circ\pi}(\pi^* \mathcal{M}_{T_Z}),$$

which implies that $\mathcal{P}_{c\circ\pi, d\circ\pi}(m_u) \in \operatorname{int} \mathcal{P}_{c\circ\pi, d\circ\pi}(\mathcal{M}_\sigma)$. We shall use the following result.

Lemma 17 ([5, Theorem 4])**.** *Assume that $\sigma \colon \Sigma_A \to \Sigma_A$ is topologically mixing and let m_u be the unique equilibrium measure for $I_u \circ \pi$. If*

$$\operatorname{span}\{c \circ \pi, d \circ \pi, \overline{I_u \circ \pi}\} \subset E_\sigma(\Sigma_A)$$

and $\mathcal{P}_{c\circ\pi, d\circ\pi}(m_u) \in \operatorname{int} \mathcal{P}_{c\circ\pi, d\circ\pi}(\mathcal{M}_\sigma)$, then

$$\dim_{I_u \circ \pi} C_\sigma = \dim_{I_u \circ \pi} \Sigma_A.$$

Moreover, it follows from Corollary 5.4 in [24] that

$$\dim_u \pi(B) = \dim_{u \circ \pi} B$$

for every set $B \subset \Sigma_A$. Hence, by Lemma 17 we obtain

$$\begin{aligned} \dim_{I_u} Z = \dim_{I_u} \pi(\Sigma_A) &= \dim_{I_u \circ \pi} \Sigma_A \\ &= \dim_{I_u \circ \pi} C_\sigma = \dim_{I_u} \pi(C_\sigma) = \dim_{I_u} C_{T_Z}, \end{aligned}$$

which yields the desired result. □

We proceed with the proof of the theorem. By Lemma 15, we have

$$\begin{aligned}\dim_u \Lambda &= \dim_u\big\{\phi_s(x) \in \Lambda : x \in Z \text{ and } s \in [0, \tau(x)]\big\}\\ &= \inf\big\{\alpha \in \mathbb{R} : N_\alpha(Z) = 0\big\} = \dim_{I_u} Z.\end{aligned}$$

It follows from Proposition 8 together with Lemmas 14 and 15 that

$$\begin{aligned}\dim_u C_\Phi &= \dim_u\big\{\phi_s(x) \in \Lambda : x \in C_{T_Z} \text{ and } s \in [0, \tau(x)]\big\}\\ &= \inf\big\{\alpha \in \mathbb{R} : N_\alpha(C_{T_Z}) = 0\big\} = \dim_{I_u} C_{T_Z}.\end{aligned}$$

Finally, by Lemma 16 we conclude that

$$\dim_u \Lambda = \dim_{I_u} Z = \dim_{I_u} C_{T_Z} = \dim_u C_\Phi.$$

This completes the proof of the theorem. □

We note that the hypothesis in Theorem 13 that $\Phi|_\Lambda$ is topologically mixing ensures that T_Z is also topologically mixing, which in its turn guarantees the uniqueness of the equilibrium measures for T_Z and for the symbolic dynamics. Moreover, the set $\operatorname{int} \mathcal{P}(\mathcal{M}_\Phi)$ is assumed to be nonempty in the theorem. It is this property that ensures that the irregular set C_Φ is nonempty and ultimately that it can have positive and in fact full u-dimension (as an application of Lemma 17 that contains a corresponding statement for symbolic dynamics).

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