# NONADDITIVE TOPOLOGICAL PRESSURE FOR FLOWS

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ABSTRACT. We introduce a version of the nonadditive topological pressure for flows and we describe some of its main properties. In particular, we discuss how the nonadditive topological pressure varies with the data and we establish a variational principle in terms of the Kolmogorov–Sinai entropy. We also consider corresponding capacity topological pressures. In the particular case of subadditive families of functions we give a simpler characterization of these pressures. To the possible extent we follow corresponding arguments for maps, although various proofs require nontrivial modifications.

## 1. INTRODUCTION

The thermodynamic formalism can be described as a rigorous study of certain mathematical structures inspired by thermodynamics. This includes the notion of the topological pressure of a continuous function, introduced by Ruelle in [14] for expansive maps and by Walters in [17] in the general case. Given a continuous map $f\colon X \to X$ on a compact metric space, the *topological pressure* of a continuous function $\varphi\colon X \to \mathbb{R}$ (with respect to $f$) is defined by

$$P(\varphi) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \sup_E \sum_{x\in E} \exp \sum_{k=0}^{n-1} \varphi(f^k(x)),$$

with the supremum taken over all $(n,\varepsilon)$-separated sets $E \subset X$. We recall that a set $E \subset X$ is said to be $(n,\varepsilon)$-separated if for any $x, y \in E$ with $x \neq y$ there exists $k \in \{0, \ldots, n-1\}$ such that

$$d(f^k(x), f^k(y)) > \varepsilon.$$

Taking $\varphi = 0$ we recover the notion of the topological entropy $h(f)$ of the map $f$ given by

$$h(f) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log N(n,\varepsilon),$$

where $N(n,\varepsilon)$ denotes the maximal cardinality of an $(n,\varepsilon)$-separated set.

The theory comprising the thermodynamic formalism and its many applications is a quite active and broad independent field, with many directions of research. For example, consider the variational principle for the topological

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pressure, established by Ruelle in [14] for expansive maps and by Walters in [17] in the general case. It says that

$$P(\varphi) = \sup_{\mu} \left( h_\mu(f) + \int_X \varphi \, d\mu \right),$$

with the supremum taken over all $f$-invariant probability measures $\mu$ on $X$ and where $h_\mu(f)$ is the Kolmogorov–Sinai entropy of $f$ with respect to $\mu$. The theory also includes a discussion of the existence and uniqueness of equilibrium and Gibbs measures, among many other properties. We recall that an $f$-invariant probability measure $\mu$ on $X$ is called an *equilibrium measure* for $\varphi$ if

$$P(\varphi) = h_\mu(f) + \int_X \varphi \, d\mu.$$

It turns out that these measures and particularly whether they possess or not the so-called Gibbs property, is crucial in the dimension theory and multifractal analysis of dynamical systems. We refer the reader to the books [4, 8, 10, 15] for details and further references, although a brief discussion is given in the following paragraphs.

One of the major applications of the thermodynamic formalism is to the dimension theory of dynamical systems, and particularly to multifractal analysis. The main objective of the dimension theory of dynamical systems is to measure the complexity of an invariant object, such as an invariant set or an invariant measure, from the dimensional point of view. This includes using topological entropy, Hausdorff dimension, box dimension and topological pressure, among many other characteristics. We refer the reader to the books [2, 12] for a detailed presentation of substantial parts of the theory.

The reason for this relation between the thermodynamic formalism and the dimension theory of dynamical systems is that the unique solution $s$ of the equation

$$P(s\varphi) = 0,$$

for some appropriate function $\varphi$, is often related to the Hausdorff dimension of a given invariant set. This equation was first considered by Bowen in [5] (see also [9, 16] for other early seminal works on the study of the dimension of repellers and hyperbolic sets). In particular, if $\mu_s$ is an equilibrium measure for $s\varphi$, then

$$P(s\varphi) = h_{\mu_s}(f) + s \int_X \varphi \, d\mu_s$$

and so

$$P(s\varphi) = 0 \quad \Leftrightarrow \quad s = -\frac{h_{\mu_s}(f)}{\int_X \varphi \, d\mu_s}.$$

This already motivates the interest in equilibrium measures in the context of dimension theory. It turns out that virtually all known equations used to compute or to estimate the dimension of an invariant set are particular cases of Bowen's equation or of some appropriate generalization.

A subfield of the dimension theory of dynamical systems is multifractal analysis, which studies the complexity of the level sets of any invariant local quantity obtained from a dynamical system. This includes Birkhoff averages, Lyapunov exponents, pointwise dimensions and local entropies, among various others. These functions are typically only measurable and so again it is appropriate to use quantities such as the topological entropy or the Hausdorff dimension to measure their complexity.

The nonadditive thermodynamic formalism was introduced in [1] as a generalization of the classical thermodynamic formalism considered above, essentially with the topological pressure $P(\varphi)$ of a continuous function $\varphi$ replaced by the topological pressure $P(\Phi)$ of a sequence of continuous functions $\Phi = (\varphi_n)_{n\in\mathbb{N}}$. Indeed, for a certain class of sequences we have

$$P(\Phi) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \sup_E \sum_{x\in E} \exp \varphi_n(x),$$

with the supremum taken over all $(n,\varepsilon)$-separated sets $E \subset X$. However, the former limit may not exist for an arbitrary sequence $\Phi$ (we refrain from discussing the details here). For this reason and since we also need to consider functions on sets that need not be compact nor invariant, for example in view of the applications to dimension theory and multifractal analysis, the pressure $P(\Phi)$ is introduced in terms of general Carathéodory dimension characteristics (see [12]).

We note that the nonadditive thermodynamic formalism contains as a particular case a new formulation of the subadditive thermodynamic formalism introduced by Falconer in [7]. For additive sequences, it recovers the notion of topological pressure introduced by Pesin and Pitskel' in [13] as well as the notions of lower and upper capacity topological pressures introduced by Pesin in [11] for an arbitrary set. Thus, the nonadditive thermodynamic formalism also plays a unifying role, besides allowing to consider general classes of sequences of functions and arbitrary sets.

The initial motivation to introduce the nonadditive thermodynamic formalism was the study of a general class of invariant sets that may lack some uniformity in their construction, for example because the law that defines a given set changes with time or because the behavior is not the same in all directions. Incidentally, these difficulties cause that in some situations, at least at the present stage of the theory, we are only able to establish dimension estimates instead of computing the dimension. Sometimes one can obtain sharp dimension estimates, which often requires a more elaborate approach, starting essentially with seminal work of Douady and Oesterlé in [6], who devised an approach to cover the set in a optimal manner.

Over the last decades, the dimension theory of dynamical systems steadily developed into an independent and quite active field of research (see for example the books [2, 3]). However, while the dimension theory and multifractal analysis for maps are quite developed, the corresponding theory for flows has experienced a slower progress. To a large extent this happens because a corresponding result for flows often requires one of two opposite approaches: in most situations either the result can be reduced to the case of maps or it requires substantial changes or even new ideas. Because of this, many parts of the theory are either only sketched or were not yet developed.

Our main aim is precisely to introduce a version of the nonadditive topological pressure for flows and to describe some of its main properties, thus paving the way for a corresponding nonadditive thermodynamic formalism.

In particular, we establish a variational principle for the nonadditive topological pressure. We also consider the particular case of subadditive families of functions for which it is possible to give a simpler characterization of the pressure. To the possible extent we follow corresponding arguments for maps, taken essentially from [1, 3, 12, 13], although various proofs require nontrivial modifications.

## 2. Nonadditive topological pressures

In this section we introduce the notion of nonadditive topological pressure for a flow (more precisely, we introduce the nonadditive topological pressure and the nonadditive lower and upper capacity topological pressures). To the possible extent we mimic the definition in the case of maps.

Let $\Phi$ be a continuous flow on a compact metric space $(X, d)$, that is, a family of homeomorphisms $\varphi_t\colon X \to X$ such that $\varphi_0 = \mathrm{id}$ and $\varphi_t \circ \varphi_s = \varphi_{t+s}$ for all $t, s \in \mathbb{R}$. Given $x \in X$ and $t, \varepsilon > 0$, we consider the set

$$B_t(x,\varepsilon) = \big\{y \in X : d(\varphi_s(y), \varphi_s(x)) < \varepsilon \text{ for } s \in [0,t]\big\}.$$

Moreover, let $a = (a_t)_{t>0}$ be a family of continuous functions $a_t : X \to \mathbb{R}$ with *tempered variation,* that is, such that

$$\lim_{\varepsilon\to 0} \overline{\lim_{t\to+\infty}} \frac{\gamma_t(a,\varepsilon)}{t} = 0, \tag{1}$$

where

$$\gamma_t(a,\varepsilon) = \sup\big\{|a_t(y) - a_t(z)| : y, z \in B_t(x,\varepsilon) \text{ for some } x \in X\big\}.$$

Now we introduce the nonadditive topological pressures. Given $\varepsilon > 0$, we say that a set $\Gamma \subset X \times \mathbb{R}_0^+$ *covers* a subset $Z \subset X$ if

$$\bigcup_{(x,t)\in\Gamma} B_t(x,\varepsilon) \supset Z \tag{2}$$

and we write

$$a(x,t,\varepsilon) = \sup\big\{a_t(y) : y \in B_t(x,\varepsilon)\big\} \quad \text{for } (x,t) \in \Gamma.$$

For each $Z \subset X$ and $\alpha \in \mathbb{R}$, let

$$M(Z,a,\alpha,\varepsilon) = \lim_{T\to+\infty} \inf_{\Gamma} \sum_{(x,t)\in\Gamma} \exp(a(x,t,\varepsilon) - \alpha t), \tag{3}$$

with the infimum taken over all countable sets $\Gamma \subset X \times [T, +\infty)$ covering $Z$, and let

$$\underline{M}(Z,a,\alpha,\varepsilon) = \underline{\lim_{T\to+\infty}} \inf_{\Gamma} \sum_{(x,t)\in\Gamma} \exp(a(x,t,\varepsilon) - \alpha t) \tag{4}$$

and

$$\overline{M}(Z,a,\alpha,\varepsilon) = \overline{\lim_{T\to+\infty}} \inf_{\Gamma} \sum_{(x,t)\in\Gamma} \exp(a(x,t,\varepsilon) - \alpha t), \tag{5}$$

with the infimum taken over all countable sets $\Gamma \subset X \times \{T\}$ covering $Z$. When $\alpha$ goes from $-\infty$ to $+\infty$, the quantities in (3), (4) and (5) jump from $+\infty$ to 0 at unique values and so one can define

$$P(a|_Z,\varepsilon) = \inf\big\{\alpha \in \mathbb{R} : M(Z,a,\alpha,\varepsilon) = 0\big\},$$

$$\underline{P}(a|_Z,\varepsilon) = \inf\big\{\alpha \in \mathbb{R} : \underline{M}(Z,a,\alpha,\varepsilon) = 0\big\},$$

$$\overline{P}(a|_Z,\varepsilon)=\inf\{\alpha\in\mathbb{R}:\overline{M}(Z,a,\alpha,\varepsilon)=0\}.$$

**Theorem 1.** *For any family of continuous functions $a$ with tempered variation and any set $Z\subset X$, the limits*

$$P(a|_Z)=\lim_{\varepsilon\to0}P(a|_Z,\varepsilon),\ \underline{P}(a|_Z)=\lim_{\varepsilon\to0}\underline{P}(a|_Z,\varepsilon),\ \overline{P}(a|_Z)=\lim_{\varepsilon\to0}\overline{P}(a|_Z,\varepsilon)$$

*exist.*

*Proof.* Take $\delta\in(0,\varepsilon)$ and $\Gamma\subset X\times\mathbb{R}_0^+$ with $Z\subset\bigcup_{(x,t)\in\Gamma}B_t(x,\delta)$. Since $B_t(x,\delta)\subset B_t(x,\varepsilon)$, property (2) also holds. Let

$$\gamma(\varepsilon)=\varlimsup_{t\to+\infty}\frac{\gamma_t(a,\varepsilon)}{t}. \tag{6}$$

Given $\eta>0$, $z\in B_t(x,\delta)$ and $y\in B_t(x,\varepsilon)$, we have

$$a_t(y)-a_t(z)\le|a_t(y)-a_t(z)|\le\gamma_t(a,\varepsilon)\le t(\gamma(\varepsilon)+\eta)$$

for any sufficiently large $t$. Thus,

$$a_t(y)\le\sup_{z\in B_t(x,\delta)}[a_t(z)+t(\gamma(\varepsilon)+\eta)]\le a(x,t,\delta)+t(\gamma(\varepsilon)+\eta)$$

and

$$a(x,t,\varepsilon)\le a(x,t,\delta)+t(\gamma(\varepsilon)+\eta)$$

for any sufficiently large $t$. Therefore,

$$M(Z,\alpha,a,\varepsilon)\le M(Z,\alpha-\gamma(\varepsilon)-\eta,a,\delta)$$

and so

$$P(a|_Z,\varepsilon)\le P(a|_Z,\delta)+\gamma(\varepsilon)+\eta.$$

Letting $\delta\to0$ we obtain

$$P(a|_Z,\varepsilon)-\gamma(\varepsilon)-\eta\le\varliminf_{\delta\to0}P(a|_Z,\delta). \tag{7}$$

By (1) we have $\gamma(\varepsilon)\to0$ when $\varepsilon\to0$, which together with the arbitrariness of $\eta$ yields the inequality

$$\varlimsup_{\varepsilon\to0}P(a|_Z,\varepsilon)\le\varliminf_{\delta\to0}P(a|_Z,\delta).$$

This shows that $P(a|_Z)$ is well defined. The existence of the other two limits can be established in a similar manner. ∎

The number $P(a|_Z)$ is called the *nonadditive topological pressure* of the family $a$ on $Z$, while $\underline{P}(a|_Z)$ and $\overline{P}(a|_Z)$ are called, respectively, the *nonadditive lower* and *upper capacity topological pressures* of $a$ on $Z$. Clearly,

$$P(a|_Z)\le\underline{P}(a|_Z)\le\overline{P}(a|_Z)$$

and if $Z_1\subset Z_2$, then

$$P(a|_{Z_1})\le P(a|_{Z_2}),\quad \underline{P}(a|_{Z_1})\le\underline{P}(a|_{Z_2}),\quad \overline{P}(a|_{Z_1})\le\overline{P}(a|_{Z_2}).$$

We will drop the prefix "nonadditive" if there is no danger of confusion.

The classical notion of topological pressure for a flow corresponds to consider a family of functions $a=(a_t)_{t>0}$ defined by

$$a_t(x)=\int_0^t b(\varphi_s(x))\,ds$$

for some continuous function $b\colon X \to \mathbb{R}$. Given $x, y \in B_t(z,\varepsilon)$, we obtain

$$\begin{aligned}|a_t(x) - a_t(y)| &= \left|\int_0^t b(\varphi_s(x))\,ds - \int_0^t b(\varphi_s(y))\,ds\right| \\ &\le \int_0^t |b(\varphi_s(x)) - b(\varphi_s(y))|\,ds \\ &\le t\sup\bigl\{|b(\varphi_s(x)) - b(\varphi_s(y))| : s \in [0,t]\bigr\} \\ &\le t\sup\bigl\{|b(\bar{x}) - b(\bar{y})| : d(\bar{x},\bar{y}) \le \varepsilon\bigr\}.\end{aligned}$$

Therefore,

$$\frac{\gamma_t(a,\varepsilon)}{t} \le \sup\bigl\{|b(\bar{x}) - b(\bar{y})| : d(\bar{x},\bar{y}) \le \varepsilon\bigr\}$$

and it follows from the uniform continuity of $b$ that property (1) holds.

## 3. Basic properties of the topological pressures

In this section we describe how the topological pressures $P(a|_Z)$, $\underline{P}(a|_Z)$ and $\overline{P}(a|_Z)$ vary with the family of functions $a$ and with the set $Z$.

We first describe the dependence of the topological pressures on $a$. For each $s > 0$, let $a^s$ be the family of functions $(a_{t+s})_{t>0}$.

**Theorem 2.** *For any family of continuous functions $a$ with tempered variation and any set $Z \subset X$, if there exists $K > 0$ such that $|a_{t+s} - a_t| < Ks$ for all $t, s > 0$, then*

$$P(a^s|_Z) = P(a|_Z), \quad \underline{P}(a^s|_Z) = \underline{P}(a|_Z), \quad \overline{P}(a^s|_Z) = \overline{P}(a|_Z)$$

*for every $s > 0$.*

*Proof.* Given $s, t, \varepsilon > 0$ and $x \in X$, let

$$r(x,s,\varepsilon) = \sup\bigl\{r \ge 0 : B(x,r) \subset B_{t+s}(x,\varepsilon)\bigr\}, \tag{8}$$

where $B(x,r)$ is the open ball of radius $r$ centered at $x$. Clearly, $r(x,s,\varepsilon) \in [0,\varepsilon]$. If $r(x,s,\varepsilon) = 0$, then there exists a sequence $y_n \in X$ converging to $x$ and a sequence $\tau_n \in [0,t+s]$ such that

$$d(\varphi_{\tau_n}(x), \varphi_{\tau_n}(y_n)) \ge \varepsilon \quad \text{for all } n.$$

Without loss of generality we may assume that $\tau_n$ converges to some number $\tau \in [0,t+s]$, which gives

$$0 = \lim_{n\to+\infty} d(\varphi_{\tau_n}(x), \varphi_{\tau_n}(y_n)) \ge \varepsilon.$$

This contradiction implies that $r(x,s,\varepsilon) > 0$. Now we show that

$$r(s,\varepsilon) := \inf\{r(x,s,\varepsilon) : x \in X\} > 0. \tag{9}$$

Otherwise it would exist a sequence $x_n \in X$ with

$$\lim_{n\to+\infty} r(x_n,s,\varepsilon) = 0. \tag{10}$$

Since $X$ is compact, we may assume that $x_n$ converges to some point $\bar{x} \in X$. Take

$$\bar{r} = r(\bar{x}, s, \varepsilon/2) \quad \text{and} \quad x_n \in B(\bar{x}, \bar{r}/2).$$

By (8), we have $x_n \in B_{t+s}(\bar{x}, \varepsilon/2)$. Moreover, if $y \in B(x_n, \bar{r}/2) \subset B(\bar{x}, \bar{r})$, then $y \in B_{t+s}(\bar{x}, \varepsilon/2)$ and

$$d(\varphi_\tau(y), \varphi_\tau(x_n)) \leq d(\varphi_\tau(y), \varphi_\tau(\bar{x})) + d(\varphi_\tau(\bar{x}), \varphi_\tau(x_n)) \leq \varepsilon$$

for every $\tau \in [0, t+s]$, which shows that $y \in B_{t+s}(x_n, \varepsilon)$. Therefore,

$$B(x_n, \bar{r}/2) \subset B_{t+s}(x_n, \varepsilon)$$

and so $r(x_n, s, \varepsilon) \geq \bar{r}/2$ for $x_n \in B(\bar{x}, \bar{r}/2)$, which contradicts to (10). This establishes property (9).

Observe that since $B_t(x, r(s, \varepsilon)) \subset B(x, r(s, \varepsilon))$ and $r(s, \varepsilon) \leq r(x, s, \varepsilon)$, by (8) we have

$$B_t(x, r(s, \varepsilon)) \subset \overline{B_{t+s}(x, \varepsilon)}.$$

Hence, for each $x \in X$ we obtain

$$\sup\bigl\{|a_{t+s}(y) - a_{t+s}(z)| : y, z \in B_{t+s}(x, \varepsilon)\bigr\} \leq \gamma_{t+s}(a, \varepsilon)$$

and so

$$\sup\bigl\{|a_t^s(y) - a_t^s(z)| : y, z \in B_t(x, r(s, \varepsilon))\bigr\} \leq \gamma_{t+s}(a, \varepsilon).$$

Therefore,

$$\gamma_t(a^s, r(s, \varepsilon)) \leq \gamma_{t+s}(a, \varepsilon).$$

Since

$$\gamma_t(a^s, \varepsilon) \leq \gamma_t(a^s, \varepsilon') \quad \text{for } \varepsilon \leq \varepsilon',$$

we obtain

$$\begin{aligned}\varlimsup_{\varepsilon\to 0} \varlimsup_{t\to+\infty} \frac{\gamma_t(a^s, \varepsilon)}{t} &= \inf_{\varepsilon \leq r(s,\varepsilon)} \varlimsup_{t\to+\infty} \frac{\gamma_t(a^s, \varepsilon)}{t} \leq \varlimsup_{t\to+\infty} \frac{\gamma_t(a^s, r(s, \varepsilon))}{t} \\ &\leq \varlimsup_{t\to+\infty} \frac{\gamma_{t+s}(a, \varepsilon)}{t} = \varlimsup_{t\to+\infty} \frac{\gamma_t(a, \varepsilon)}{t}\end{aligned}$$

and so the family $a^s$ satisfies property (1). On the other hand, we have

$$e^{-Ks} \leq \frac{\sum_{(x,t)\in\Gamma} \exp(\sup_{B_t(x,\varepsilon)} a_{t+s} - \alpha t)}{\sum_{(x,t)\in\Gamma} \exp(a(x, t, \varepsilon) - \alpha t)} \leq e^{Ks}$$

for any countable set $\Gamma \subset X \times \mathbb{R}_0^+$ covering $Z$ and so it follows from (3) that

$$M(Z, \alpha, a, \varepsilon)e^{-Ks} \leq M(Z, \alpha, a^s, \varepsilon) \leq M(Z, \alpha, a, \varepsilon)e^{Ks}.$$

Hence, $P(a^s|_Z, \varepsilon) = P(a|_Z, \varepsilon)$ for every $\varepsilon > 0$, which implies that $P(a^s|_Z) = P(a|_Z)$. The remaining identities can be obtained in a similar manner. □

In order to describe the continuity of the topological pressures on $a$, let

$$\|a\| = \varlimsup_{t\to+\infty} \frac{1}{t} \sup\{|a_t(x)| : x \in X\}.$$

**Theorem 3.** *For any families of continuous functions $a$ and $b$ with tempered variation and any set $Z \subset X$, when the topological pressures are finite we have*

$$\begin{aligned}|P(a|_Z) - P(b|_Z)| &\leq \|a - b\|, \\ |\underline{P}(a|_Z) - \underline{P}(b|_Z)| &\leq \|a - b\|, \\ |\overline{P}(a|_Z) - \overline{P}(b|_Z)| &\leq \|a - b\|.\end{aligned}$$

*Proof.* Given $\eta > 0$, we have

$$|a_t(x) - b_t(x)| \le t(\|a - b\| + \eta)$$

for any sufficiently large $t$. Hence, it follows from the definitions that

$$M(Z, b, \alpha + \|a - b\| + \eta, \varepsilon) \le M(Z, a, \alpha, \varepsilon) \le M(Z, b, \alpha - \|a - b\| - \eta, \varepsilon)$$

and so

$$P(b|_Z, \varepsilon) - \|a - b\| - \eta \le P(a|_Z, \varepsilon) \le P(b|_Z, \varepsilon) + \|a - b\| + \eta.$$

Since $\eta$ is arbitrary, we obtain

$$|P(a|_Z, \varepsilon) - P(b|_Z, \varepsilon)| \le \|a - b\|,$$

which yields the first inequality in the theorem. The remaining inequalities can be established in a similar manner. ☐

Now we describe the dependence of the topological pressures on the set $Z$.

**Theorem 4.** *Given a family of continuous functions $a$ with tempered variation and a set $Z \subset X$, for any finite or countable union $Z = \bigcup_{i \in I} Z_i$ the following properties hold:*

1. $P(a|_Z) = \sup_{i \in I} P(a|_{Z_i})$*;*
2. $\overline{P}(a|_Z) \ge \sup_{i \in I} \overline{P}(a|_{Z_i})$*, with equality when $I$ is finite.*
3. $\underline{P}(a|_Z) \ge \sup_{i \in I} \underline{P}(a|_{Z_i})$*, with equality when $I$ is finite and $\underline{P}(a|_{Z_i}) = \overline{P}(a|_{Z_i})$ for each $i \in I$;*

*Proof.* 1. Since $Z_i \subset Z$, we have $P(a|_{Z_i}) \le P(a|_Z)$ for each $i$ and so

$$\sup_{i \in I} P(a|_{Z_i}) \le P(a|_Z). \tag{11}$$

Now take $\alpha > \sup_{i \in I} P(a|_{Z_i}, \varepsilon)$. Then $M(Z_i, a, \alpha, \varepsilon) = 0$ for each $i$. Hence, given $\delta > 0$ and $T > 0$, for each $i$ there exists $\Gamma_i \subset X \times [T, +\infty)$ covering $Z_i$ such that

$$\sum_{(x,t) \in \Gamma_i} \exp(a(x, t, \varepsilon) - \alpha t) < \frac{\delta}{2^i}.$$

Then $\Gamma = \bigcup_{i \in I} \Gamma_i$ covers $Z$ and

$$\sum_{(x,t) \in \Gamma} \exp(a(x, t, \varepsilon) - \alpha t) \le \sum_{i \in I} \sum_{(x,t) \in \Gamma_i} \exp(a(x, t, \varepsilon) - \alpha t) \le \sum_{i \in I} \frac{\delta}{2^i} \le \delta.$$

Letting $T \to +\infty$ gives $M(Z, a, \alpha, \varepsilon) \le \delta$ and so $M(Z, a, \alpha, \varepsilon) = 0$ since $\delta$ is arbitrary. Therefore, $\alpha \ge P(a|_Z, \varepsilon)$ and letting $\alpha \to \sup_{i \in I} P(a|_{Z_i}, \varepsilon)$ gives

$$\sup_{i \in I} P(a|_{Z_i}, \varepsilon) \ge P(a|_Z, \varepsilon). \tag{12}$$

On the other hand, it follows from (7) with $\gamma(\varepsilon)$ as in (6) that

$$P(a|_{Z_i}) \ge P(a|_{Z_i}, \varepsilon) - \gamma(\varepsilon).$$

Hence, by (11) and (12) we obtain

$$P(a|_Z) \ge \sup_{i \in I} P(a|_{Z_i}) \ge \sup_{i \in I} P(a|_{Z_i}, \varepsilon) - \gamma(\varepsilon) \ge P(a|_Z, \varepsilon) - \gamma(\varepsilon)$$

and the desired result follows from the fact that $\gamma(\varepsilon) \to 0$ when $\varepsilon \to 0$.

2. Since $Z_i \subset Z$, we have $\overline{P}(a|_{Z_i}) \le \overline{P}(a|_Z)$ for each $i$ and so

$$\sup_{i\in I} \overline{P}(a|_{Z_i}) \le \overline{P}(a|_Z).$$

Now assume that $I$ is finite. We need to show that

$$\max_{i\in I} \overline{P}(a|_{Z_i}) \ge \overline{P}(a|_Z). \tag{13}$$

For each $i$ consider a set $\Gamma_i \subset X \times \{T\}$ covering $Z_i$. Then $\Gamma = \bigcup_{i\in I} \Gamma_i$ covers the union $Z$ and

$$\sum_{(x,t)\in\Gamma} \exp(a(x,t,\varepsilon) - \alpha t) \le \sum_{i\in I} \sum_{(x,t)\in\Gamma_i} \exp(a(x,t,\varepsilon) - \alpha t),$$

which implies that

$$\overline{M}(Z,a,\alpha,\varepsilon) \le \sum_{i\in I} \overline{M}(Z_i,a,\alpha,\varepsilon) \le \operatorname{card} I \max_{i\in I} \overline{M}(Z_i,a,\alpha,\varepsilon).$$

Therefore,

$$\overline{P}(a|_Z,\varepsilon) \le \max_{i\in I} \overline{P}(a|_{Z_i},\varepsilon)$$

and taking the limit when $\varepsilon \to 0$ yields inequality (13).

3. Since $Z_i \subset Z$, we have $\overline{P}(a|_{Z_i}) \le \overline{P}(a|_Z)$ for each $i$ and so

$$\sup_{i\in I} \underline{P}(a|_{Z_i}) \le \underline{P}(a|_Z).$$

When $I$ is finite, by the former property we have $\max_{i\in I} \overline{P}(a|_{Z_i}) = \overline{P}(a|_Z)$. Since $\underline{P}(a|_{Z_i}) = \overline{P}(a|_{Z_i})$ for each $i$, we obtain

$$\underline{P}(a|_Z) \ge \max_{i\in I} \underline{P}(a|_{Z_i}) = \max_{i\in I} \overline{P}(a|_{Z_i}) = \overline{P}(a|_Z) \ge \underline{P}(a|_Z),$$

which shows that $\underline{P}(a|_Z) = \max_{i\in I} \underline{P}(a|_{Z_i})$ as desired. ☐

## 4. Characterizations of the capacity topological pressures

In this section we establish several alternative formulas for the lower and upper capacity topological pressures. In particular, we obtain formulas in terms of partition functions and separated sets.

### 4.1. General case.

We first describe a characterization in terms of partition functions. Given $\varepsilon, t > 0$ and $Z \subset X$, consider the *partition function*

$$\mathcal{Z}_t(Z,a,\varepsilon) = \inf_{\Gamma} \sum_{(x,t)\in\Gamma} \exp a(x,t,\varepsilon),$$

with the infimum taken over all countable sets $\Gamma \subset X \times \{t\}$ covering $Z$.

**Theorem 5.** *Given a family of continuous functions $a$ with tempered variation and a set $Z \subset X$, for each $\varepsilon > 0$ we have*

$$\underline{P}(a|_Z,\varepsilon) = \varliminf_{t\to+\infty} \frac{1}{t} \log \mathcal{Z}_t(Z,a,\varepsilon) \tag{14}$$

*and*

$$\overline{P}(a|_Z,\varepsilon) = \varlimsup_{t\to+\infty} \frac{1}{t} \log \mathcal{Z}_t(Z,a,\varepsilon). \tag{15}$$

*Proof.* Note that

$$\underline{M}(Z,a,\alpha,\varepsilon) = \varliminf_{t\to+\infty} \big(e^{-\alpha t}\mathcal{Z}_t(Z,a,\varepsilon)\big)$$

and

$$\overline{M}(Z,a,\alpha,\varepsilon) = \varlimsup_{t\to+\infty} \big(e^{-\alpha t}\mathcal{Z}_t(Z,a,\varepsilon)\big).$$

Given $\alpha > \underline{P}(a|_Z,\varepsilon)$, there exists a sequence $t_n \nearrow +\infty$ such that

$$e^{-\alpha t_n}\mathcal{Z}_{t_n}(Z,a,\varepsilon) < 1 \quad \text{for } n \in \mathbb{N}.$$

Therefore, $\log \mathcal{Z}_{t_n}(Z,a,\varepsilon) < \alpha t_n$ and so

$$\varliminf_{t\to+\infty} \frac{1}{t} \log \mathcal{Z}_t(Z,a,\varepsilon) \le \underline{P}(a|_Z,\varepsilon). \tag{16}$$

On the other hand, for each $\alpha < \underline{P}(a|_Z,\varepsilon)$ we have $e^{-\alpha t}\mathcal{Z}_t(Z,a,\varepsilon) > 1$ for any sufficiently large $t$ and so

$$\varliminf_{t\to+\infty} \frac{1}{t} \log \mathcal{Z}_t(Z,a,\varepsilon) \ge \alpha.$$

Therefore,

$$\varliminf_{t\to+\infty} \frac{1}{t} \log \mathcal{Z}_t(Z,a,\varepsilon) \ge \underline{P}(a|_Z,\varepsilon),$$

which together with (16) establishes property (14). The second property can be obtained in a similar manner. ☐

Now we establish additional formulas for the lower and upper capacity topological pressures in terms of separated sets and in terms of certain covers. For each $t > 0$ consider the distance $d_t$ on $X$ defined by

$$d_t(x,y) = \max\big\{d(\varphi_\tau(x),\varphi_\tau(y)) : \tau \in [0,t]\big\}.$$

Given $\varepsilon > 0$, a set $E \subset X$ is said to be $(t,\varepsilon)$-*separated* if $d_t(x,y) > \varepsilon$ for any $x,y \in E$ with $x \ne y$. Given $Z \subset X$ and $\varepsilon > 0$, for each $t > 0$ let

$$\mathcal{R}_t(Z,a,\varepsilon) = \sup_E \sum_{x\in E\cap Z} \exp a_t(x),$$

with the supremum taken over all $(t,\varepsilon)$-separated sets $E \subset X$. Moreover, let

$$\mathcal{S}_t(Z,a,\varepsilon) = \inf_{\mathcal{V}} \sum_{V\in\mathcal{V}} \exp\sup_V a_t,$$

with the infimum taken over all finite open covers $\mathcal{V}$ of $Z$ by sets $V$ with

$$\sup\big\{d_t(x,y) : x,y \in V\big\} < \varepsilon. \tag{17}$$

**Theorem 6.** *For any family of continuous functions $a$ with tempered variation and any set $Z \subset X$, we have*

$$\underline{P}(a|_Z) = \lim_{\varepsilon\to0} \varliminf_{t\to+\infty} \frac{1}{t} \log \mathcal{R}_t(Z,a,\varepsilon) = \lim_{\varepsilon\to0} \varliminf_{t\to+\infty} \frac{1}{t} \log \mathcal{S}_t(Z,a,\varepsilon) \tag{18}$$

*and*

$$\overline{P}(a|_Z) = \lim_{\varepsilon\to0} \varlimsup_{t\to+\infty} \frac{1}{t} \log \mathcal{R}_t(Z,a,\varepsilon) = \lim_{\varepsilon\to0} \varlimsup_{t\to+\infty} \frac{1}{t} \log \mathcal{S}_t(Z,a,\varepsilon). \tag{19}$$

*Proof.* We first show that

$$\mathcal{R}_t(Z,a,2\varepsilon) \le \mathcal{Z}_t(Z,a,\varepsilon) \le \mathcal{R}_t(Z,a,\varepsilon)e^{\gamma_t(a,\varepsilon)}. \tag{20}$$

Note that distinct elements of $Z$ in a $(t,2\varepsilon)$-separated set $E$ belong to distinct elements of any given open cover $\{B_t(x,\varepsilon) : (x,t) \in \Gamma\}$ of $Z$. Hence, for each $x \in E \cap Z$ we have $a_t(x) \le a(y_x,t,\varepsilon)$ for some $(y_x,t) \in \Gamma$ with $x \in B_t(y_x,\varepsilon)$. Therefore,

$$\sum_{x\in E\cap Z} \exp a_t(x) \le \sum_{x\in E\cap Z} \exp a(y_x,t,\varepsilon) \le \sum_{(y,t)\in\Gamma} \exp a(y,t,\varepsilon).$$

This establishes the first inequality in (20). Now let $E$ be a $(t,\varepsilon)$-separated set such that for each $z \in Z$ the union $E \cup \{z\}$ is not a $(t,\varepsilon)$-separated set. Then $E \times \{t\}$ covers $Z$. Therefore,

$$\begin{aligned}\inf_\Gamma \sum_{(x,t)\in\Gamma} \exp \inf_{y\in B_t(x,\varepsilon)} a_t(y) &\le \sum_{(x,t)\in(E\cap Z)\times\{t\}} \exp \inf_{y\in B_t(x,\varepsilon)} a_t(y)\\ &\le \sum_{x\in E\cap Z} \exp a_t(x) \le \mathcal{R}_t(Z,a,\varepsilon),\end{aligned} \tag{21}$$

with the first infimum taken over all $\Gamma \subset X \times \{t\}$ covering $Z$. If $y,z \in B_t(x,\varepsilon)$, then $a_t(y) \ge a_t(z) - \gamma_t(a,\varepsilon)$ and so

$$\inf_{y\in B_t(x,\varepsilon)} a_t(y) \ge \sup_{z\in B_t(x,\varepsilon)} a_t(z) - \gamma_t(a,\varepsilon).$$

Hence, it follows from (21) that the second inequality in (20) also holds.

Now we show that

$$\mathcal{S}_t(Z,a,2\varepsilon) \le \mathcal{Z}_t(Z,a,\varepsilon) \le \mathcal{S}_t(Z,a,\varepsilon). \tag{22}$$

The first inequality follows from the fact that $B_t(x,\varepsilon)$ has $d_t$-diameter less than $2\varepsilon$. For the second inequality, we observe that if $\mathcal{V}$ is a finite open cover of $Z$ by sets $V$ satisfying (17), then $Z \subset \bigcup_{V\in\mathcal{V}} B_t(x_V,\varepsilon)$ for every $x_V \in V$ since $V \subset B_t(x_V,\varepsilon)$. Therefore,

$$\sum_{V\in\mathcal{V}} \exp a(x,t,\varepsilon) \ge \mathcal{Z}_t(Z,a,\varepsilon),$$

which yields the second inequality in (22).

By (20) and (22) we have

$$\begin{aligned}\mathcal{Z}_t(Z,a,\varepsilon) &\le \mathcal{S}_t(Z,a,\varepsilon) \le \mathcal{Z}_t\left(Z,a,\frac{\varepsilon}{2}\right)\\ &\le \mathcal{R}_t\left(Z,a,\frac{\varepsilon}{2}\right)e^{\gamma_t(a,\varepsilon/2)} \le \mathcal{Z}_t\left(Z,a,\frac{\varepsilon}{4}\right)e^{\gamma_t(a,\varepsilon/2)}.\end{aligned}$$

The formulas in (18) and (19) follow readily from these inequalities together with (14) and (15). ☐

4.2. **Subadditive case.** In this section we consider the particular case of subadditive families of functions and we describe additional characterizations of the nonadditive topological pressures in this case. A family of functions $a = (a_t)_{t>0}$ is said to be *subadditive* if

$$a_{s+t}(x) \le a_s(x) + a_t(\varphi_s(x))$$

for all $s, t > 0$ and $x \in X$. Recall that a set $A \subset X$ is said to be $\Phi$-*invariant* if $\varphi_t(A) = A$ for all $t \in \mathbb{R}$.

**Theorem 7.** *For any subadditive family of continuous functions $a$ with tempered variation and any $\Phi$-invariant set $Z \subset X$, we have*

$$\hat{P}(a|_Z) := \underline{P}(a|_Z) = \overline{P}(a|_Z) = \lim_{\varepsilon \to 0} \lim_{t \to +\infty} \frac{1}{t} \log \mathcal{S}_t(Z, a, \varepsilon). \tag{23}$$

*If, in addition, $Z$ is compact, then $P(a|_Z) = \hat{P}(a|_Z)$.*

*Proof.* We first show that the limit on the right-hand side of (23) exists. We shall always assume in what follows that $Z$ is $\Phi$-invariant.

**Lemma 8.** *For each $i \in \mathbb{N}$ let $\mathcal{V}_i$ be a cover of $Z$ by sets of $d_{t_i}$-diameter less than $\varepsilon$. Then*

$$\mathcal{V}^n = \left\{ \bigcap_{i=1}^{n} \varphi_{-\tau_i} V_i : V_i \in \mathcal{V}_i \right\} \tag{24}$$

*is a cover of $Z$ by sets of $d_{\tau_{n+1}}$-diameter less than $\varepsilon$, where $\tau_n = \sum_{j=0}^{n-1} t_j$ taking $t_0 = 0$. Moreover,*

$$\sup \left\{ a_{\tau_{n+1}}(x) : x \in \bigcap_{i=1}^{n} \varphi_{-\tau_i} V_i \right\} \le \sum_{i=1}^{n} \sup_{x \in V_i} a_{t_i}(x). \tag{25}$$

*Proof of the lemma.* Since $Z$ is $\Phi$-invariant, the family $\mathcal{V}^n$ is a cover. Now we proceed by induction on $n$. For $n = 1$ the statement is clear. For $n > 1$ take $\tau \in [0, \tau_{n+1}]$ and

$$x, y \in V = \bigcap_{i=1}^{n} \varphi_{-\tau_i} V_i.$$

If $V' = \bigcap_{i=1}^{n-1} \varphi_{-\tau_i} V_i$ has $d_{\tau_n}$-diameter less than $\varepsilon$, then

$$d(\varphi_\tau(x), \varphi_\tau(y)) < \varepsilon \quad \text{for } \tau \in [0, \tau_n]$$

since $V \subset V'$. Moreover, $x, y \in \varphi_{-\tau_n} V_n$ and so $\varphi_{\tau_n}(x), \varphi_{\tau_n}(y) \in V_n$ and

$$d(\varphi_\tau(\varphi_{\tau_n}(x)), \varphi_\tau(\varphi_{\tau_{n-1}}(y))) < \varepsilon$$

for $\tau \in [0, t_n]$. This shows that

$$d(\varphi_\tau(x), \varphi_\tau(y)) < \varepsilon \quad \text{for } \tau \in [\tau_n, \tau_{n+1}].$$

Therefore, the set $V$ has $d_{\tau_{n+1}}$-diameter less than $\varepsilon$. Moreover, since $a$ is subadditive, for each $x \in V$ we obtain

$$a_{\tau_{n+1}}(x) \le a_{\tau_n}(x) + a_{t_n}(\varphi_{\tau_n}(x)) \le \sum_{i=1}^{n-1} \sup_{x \in V_i} a_{t_i}(x) + \sup_{x \in V_n} a_{t_n}(x)$$

because $V \subset \varphi_{-\tau_n} V_n$ and so $\varphi_{\tau_n}(x) \in V_n$. □

One can now establish the existence of the limit in (23). By Lemma 8, if $\mathcal{V}_1$ is a cover of $Z$ by sets of $d_s$-diameter less than $\varepsilon$ and $\mathcal{V}_2$ is a cover of $Z$ by sets of $d_t$-diameter less than $\varepsilon$, then $\mathcal{V}^2$ is a cover of $Z$ by sets of $d_{s+t}$-diameter less than $\varepsilon$ and

$$\sup_{x \in V_1 \cap \varphi_{-s} V_2} a_{s+t}(x) \le \sup_{x \in V_1} a_s(x) + \sup_{x \in V_2} a_t(x).$$

Therefore,

$$\mathcal{S}_{t+s}(Z,a,\varepsilon) \le \sum_{V_1\in\mathcal{V}_1} \exp\sup_{x\in V_1} a_t(x) \times \sum_{V_2\in\mathcal{V}_2} \exp\sup_{x\in V_2} a_s(x)$$

and so

$$\log \mathcal{S}_{t+s}(Z,a,\varepsilon) \le \log \mathcal{S}_t(Z,a,\varepsilon) + \log \mathcal{S}_s(Z,a,\varepsilon).$$

This readily implies that the limit

$$\lim_{t\to+\infty} \frac{1}{t} \log \mathcal{S}_t(Z,a,\varepsilon)$$

exists. Together with Theorem 6 this yields property (23).

Now assume that $Z$ is compact. To establish the last property, take $\varepsilon > 0$ and $\alpha > P(a|_Z, \varepsilon/2)$. There exist $s > 0$ and a countable set $\Gamma \subset X \times [s, +\infty)$ covering $Z$ such that

$$N(\mathcal{V}) := \sum_{V\in\mathcal{V}} \exp\left(\sup_{z\in V} a_t(z) - \alpha t\right) < 1,$$

where

$$\mathcal{V} = \big\{B_t(x,\varepsilon/2) : (x,t)\in\Gamma\big\}.$$

Note that the sets $B_t(x,\varepsilon/2)$ have $d_t$-diameter less than $\varepsilon$. Since $Z$ is compact, we may assume that $\mathcal{V}$ and so also $\Gamma$ are finite. Let $T = \max_{(x,t)\in\Gamma} t$. By Lemma 8, the family $\mathcal{V}^n$ in (24) covers $Z$ and (25) holds. Therefore,

$$N(\mathcal{V}^n) \le \prod_{i=1}^{n} \sum_{V\in\mathcal{V}} \exp\left(\sup_{x\in V} a_{t(V)}(x) - \alpha t(V)\right) = N(\mathcal{V})^n, \tag{26}$$

where $t(V) = t$ for $V = B_t(x,\varepsilon/2)$. Now we consider the cover

$$\mathcal{V}^\infty = \big\{V : V \in \mathcal{V}^n \text{ for some } n\in\mathbb{N}\big\}.$$

By (26) we have

$$N(\mathcal{V}^\infty) \le \sum_{n=1}^{+\infty} N(\mathcal{V}^n) \le \sum_{n=1}^{+\infty} N(\mathcal{V})^n < +\infty.$$

Given $S \ge 2T$, let $\Lambda_S \subset \mathcal{V}^\infty$ be the family of all sets $U = \bigcap_{i=1}^k \varphi_{-\tau_i} V_i$ such that $V_i = B_{t_i}(x_i, \varepsilon/2) \in \mathcal{V}$ for $i = 1, \ldots, k$ and

$$\tau_k < S - T \le \tau_{k+1}.$$

Note that $\Lambda_S$ covers $Z$. Indeed, given $z \in Z$, there exists

$$V^k = \bigcap_{i=1}^{k} \varphi_{-\tau_i} V_i \in \mathcal{V}^\infty$$

containing $z$ with $\tau_{k+1} > S$. But the point $z$ is also in $V^i$ for $i < k$ such that $\tau_i < S - T \le \tau_{i+1}$ and any such a $V^i$ belongs to $\Lambda_S$. Now write $t(U) = \tau_{i+1}$. Clearly,

$$S - T \le t(U) < S \quad \text{for } U \in \Lambda_S.$$

Finally, we consider the cover of $Z$ defined by

$$\Lambda_S' = \big\{U \cap \varphi_{-t(U)} B_{S-t(U)}(x,\varepsilon/2) : U \in \Lambda_S, (x,T)\in\Gamma'\big\},$$

where $\Gamma' \subset X \times \{T\}$ is some finite set covering $Z$. Note that any $V \in \Lambda'_S$ has $d_S$-diameter less than $\varepsilon$ and $t(V) = S$. By Lemma 8 we obtain

$$\begin{aligned}
N(\Lambda'_S) &= \sum_{U \in \Lambda_S, (x,T) \in \Gamma'} \exp\left( \sup_{z \in U \cap \varphi_{-t(U)} B_{S-t(U)}(x,\varepsilon/2)} a_S(z) - \alpha S \right) \\
&\le \sum_{U \in \Lambda_S} \exp\left( \sup_{z \in U} a_{t(U)}(z) - \alpha t(U) \right) \\
&\quad \times \sum_{(x,T) \in \Gamma'} \exp\big( a(x, S - t(U), \varepsilon/2) - \alpha(S - t(U)) \big) \\
&\le N(\mathcal{V}^\infty) \max\{1, e^{-\alpha T}\} \sum_{(x,T) \in \Gamma'} \exp(a(x, T, \varepsilon/2) - \alpha T) < +\infty.
\end{aligned}$$

Letting $S \to +\infty$ gives $\overline{M}(Z, a, \alpha, \varepsilon/2) = 0$ and so $\alpha \ge \overline{P}(a|_Z, \varepsilon/2)$. Therefore, letting $\alpha \to P(a|_Z, \varepsilon/2)$ we obtain

$$P(a|_Z, \varepsilon/2) \ge \overline{P}(a|_Z, \varepsilon/2),$$

which yields the desired property. □

## 5. Variational principle

In this section we establish a nonadditive version of the variational principle for the topological pressure for flows. To the possible extent we follow the approach in [1] in the case of maps (see [3] for additional details).

We continue to assume that $\Phi$ is a continuous flow on a compact metric space $X$. Let $\mathcal{M}$ be the set of $\Phi$-invariant probability measures on $X$, that is, the probability measures $\mu$ on $X$ such that

$$\mu(\varphi_t(A)) = \mu(A)$$

for any Borel set $A \subset X$ and any $t \in \mathbb{R}$. A measure $\mu \in \mathcal{M}$ is said to be *ergodic* for the flow $\Phi$ if every Borel $\Phi$-invariant set $A \subset X$ satisfies $\mu(A) = 0$ or $\mu(X \setminus A) = 0$. We note that in general an invariant measure for the time-1 map $\varphi_1$ need not be invariant for the flow and that an ergodic measure for the flow need not be ergodic for the time-1 map. Given a Borel $\Phi$-invariant set $Z \subset X$, let $\mathcal{M}_Z$ be the set of measures $\mu \in \mathcal{M}$ concentrated on $Z$, that is, such that $\mu(Z) = 1$.

Given $x \in X$ and $t > 0$, we define a probability measure on $X$ by

$$\mu_{x,t} = \frac{1}{t} \int_0^t \delta_{\varphi_s(x)} \, ds,$$

where $\delta_y$ is the probability measure concentrated on $\{y\}$. Let $V(x)$ be the set of weak sublimits of the family $\{\mu_{x,t}\}_{t>0}$. Then $\varnothing \ne V(x) \subset \mathcal{M}$. For each $\mu \in \mathcal{M}$ we consider the Borel $\Phi$-invariant sets

$$\mathcal{L}(Z) = \big\{x \in Z : V(x) \cap \mathcal{M}_Z \ne \varnothing\big\} \quad \text{and} \quad Z_\mu = \big\{x \in Z : V(x) = \{\mu\}\big\}.$$

The following result establishes a variational principle for the nonadditive topological pressure. For each $\mu \in \mathcal{M}$, let $h_\mu(\Phi)$ be the Kolmogorov–Sinai entropy of $\Phi$ with respect to $\mu$.

**Theorem 9.** *Let $a$ be a family of continuous functions with tempered variation such that* $\sup_{t\in[0,T]} \|a_t\|_\infty < +\infty$ *for all* $T > 0$ *and let* $Z \subset X$ *be a Borel* $\Phi$*-invariant set. If there exists a continuous function* $b\colon X \to \mathbb{R}$ *such that*

$$a_{t+s} - a_t \circ \varphi_s \to \int_0^s (b \circ \varphi_u)\, du \tag{27}$$

*uniformly on* $Z$ *when* $t \to +\infty$ *for some* $s > 0$*, then*

$$P(a|_{\mathcal{L}(Z)}) = \sup\left\{h_\mu(\Phi) + \int_Z b\, d\mu : \mu \in \mathcal{M}_Z\right\}.$$

*Proof.* We divide the proof into various steps.

*Step 1. An auxiliary lemma.* Take $x \in \mathcal{L}(Z)$ and $\mu \in V(x) \cap \mathcal{M}_Z$. Given $\delta > 0$, there exists an increasing sequence $(t_j)_{j\in\mathbb{N}}$ in $\mathbb{R}^+$ such that

$$\left|\frac{1}{t_j}\int_0^{t_j} b(\varphi_s(x))\, ds - \int_Z b\, d\mu\right| < \delta$$

for all $j \in \mathbb{N}$. This implies that

$$\left|\frac{a_{t_j}(x)}{t_j} - \int_Z b\, d\mu\right| \le \left|\frac{a_{t_j}(x)}{t_j} - \frac{1}{t_j}\int_0^{t_j} b(\varphi_u(x))\, du\right| + \delta. \tag{28}$$

Moreover, let

$$b_t = a_{t+s} - a_t \circ \varphi_s - \int_0^s (b \circ \varphi_u)\, du.$$

For each $n \in \mathbb{N}$ with $t - ns \ge 0$ we have

$$\begin{aligned} a_t - \int_0^t (b \circ \varphi_u)\, du &= a_t - a_{t-s} \circ \varphi_s - \int_0^s (b \circ \varphi_u)\, du \\ &\quad + a_{t-s} \circ \varphi_s - \int_s^t (b \circ \varphi_u)\, du \\ &= b_{t-s} + \left[a_{t-s} - \int_0^{t-s} (b \circ \varphi_u)\, du\right] \circ \varphi_s \\ &= b_{t-s} + b_{t-2s} \circ \varphi_s + \left[a_{t-2s} - \int_0^{t-2s} (b \circ \varphi_u)\, du\right] \circ \varphi_{2s} \end{aligned}$$

and so, proceeding inductively,

$$\begin{aligned} a_t - \int_0^t (b \circ \varphi_u)\, du &= \sum_{k=1}^n b_{t-ks} \circ \varphi_{(k-1)s} \\ &\quad + a_{t-ns} \circ \varphi_{ns} - \int_0^{t-ns} (b \circ \varphi_{u+ns})\, du. \end{aligned} \tag{29}$$

Hence, it follows from (28) that

$$\begin{aligned} \left|\frac{a_{t_j}(x)}{t_j} - \int_Z b\, d\mu\right| &\le \left|\frac{a_{t_j}(x)}{t_j} - \frac{1}{t_j}\int_0^{t_j} b(\varphi_u(x))\, du\right| + \delta \\ &\le \frac{1}{t_j}\sum_{k=1}^n \|b_{t_j-ks}\|_\infty + \frac{\|a_{t_j-ns}\|_\infty + (t_j - ns)\|b\|_\infty}{t_j} + \delta. \end{aligned}$$

Now let $n_j = \lfloor t_j/s \rfloor$. Then $t_j - n_j s \le s$ and since $\sup_{t\in[0,s]} \|a_t\|_\infty < +\infty$, we have

$$\frac{\|a_{t_j - n_j s}\|_\infty + (t_j - n_j s)\|b\|_\infty}{t_j} < \delta$$

for any sufficiently large $j$. Hence, by (27) and since $\sup_{t\in[0,T]} \|a_t\|_\infty < +\infty$ for all $T > 0$, taking $n = n_j$ we obtain

$$\left| \frac{a_{t_j}(x)}{t_j} - \int_Z b\, d\mu \right| \le \frac{1}{t_j} \sum_{k=1}^{n_j} \|b_{t_j - ks}\|_\infty + 2\delta \le 3\delta,$$

again for any sufficiently large $j$.

Now let $E$ be a finite set. Given $k \in \mathbb{N}$ and $c = (c_1, \ldots, c_k) \in E^k$, we define a probability measure $\mu_c$ on $E$ by

$$\mu_c(e) = \frac{1}{k} \operatorname{card}\{j : c_j = e\} \quad \text{for } e \in E.$$

Moreover, let

$$H(c) = -\sum_{e\in E} \mu_c(e) \log \mu_c(e),$$

with the convention that $0 \log 0 = 0$. Using the former observations and proceeding as in the proof of [3, Lemma 4.3.2] now for the map $\varphi_1$, we obtain the following result (recall that $h_\mu(\Phi) = h_\mu(\varphi_1)$).

**Lemma 10.** *Let $\Gamma \subset X \times \{1\}$ be a finite cover of $X$. For the open cover $\mathcal{V} = \{V_1, \ldots, V_r\}$ of $X$, where $V_j = B_1(x_j, \varepsilon/2)$ with $(x_j, 1) \in \Gamma$, there exist $m, p \in \mathbb{N}$, with $p$ arbitrary large, and a sequence $U = V_{i_1} \cdots V_{i_p}$ such that:*

1. *$x \in \bigcap_{r=1}^p \varphi_{-r+1} V_{i_r}$ and*
$$a_p(x) \le p \left( \int_Z b\, d\mu + 3\delta \right);$$
2. *there exists a subset $V \in (\mathcal{V}^m)^k$ of $U$ of length $km \ge p - m$ such that $H(V) \le m(h_\mu(\Phi) + \delta)$.*

*Step 2. Upper bound for the topological pressure.* Here we use Lemma 10 to obtain an upper bound for the topological pressure.

Given $m \in \mathbb{N}$ and $u \in \mathbb{R}$, let $Z_{m,u}$ be the set of points $x \in \mathcal{L}(Z)$ such that the two properties in Lemma 10 hold for some measure $\mu \in V(x) \cap \mathcal{M}_Z$ with

$$\int_Z b\, d\mu \in [u - \delta, u + \delta].$$

Moreover, let $n_p$ be the number of all sequences $U \in \mathcal{V}^p$ satisfying the same two properties for some $x \in Z_{m,u}$. Proceeding as in [3, Lemma 4.3.3] one can show that

$$n_p \le \exp[p(h_\mu(\Phi|_Z) + 2\delta)] = \exp[p(h_\mu(\Phi) + 2\delta)]$$

for any sufficiently large $p$ (since $\mu(Z) = 1$).

We proceed with the proof of the theorem. For each $\tau \in \mathbb{N}$, the collection of all sequences $U \in \mathcal{V}^p$ satisfying the two properties in Lemma 10 for some

$x \in Z_{m,u}$ and $p \ge \tau$ cover the set $Z_{m,u}$. Therefore,

$$\begin{aligned} &M(Z_{m,u}, a, \alpha, \varepsilon) \\ &\le \varlimsup_{\tau \to +\infty} \sum_{p=\tau}^{+\infty} n_p \exp\left[-\alpha p + p\left(\int_Z b\,d\mu + 3\delta\right) + \gamma_p(a,\varepsilon)\right] \\ &\le \varlimsup_{\tau \to +\infty} \sum_{p=\tau}^{+\infty} \exp\left[p\left(h_\mu(\Phi) + \int_Z b\,d\mu + 5\delta - \alpha + \varlimsup_{t\to+\infty} \frac{\gamma_t(a,\varepsilon)}{t}\right)\right] \\ &\le \varlimsup_{\tau \to +\infty} \sum_{p=\tau}^{+\infty} \beta^p, \end{aligned} \tag{30}$$

where

$$\beta = \exp\left(-\alpha + c + 5\delta + \varlimsup_{t\to+\infty} \frac{\gamma_t(a,\varepsilon)}{t}\right)$$

and

$$c = \sup\left\{h_\mu(\Phi) + \int_Z b\,d\mu : \mu \in \mathcal{M}_Z\right\}.$$

For

$$\alpha > c + 5\delta + \varlimsup_{t\to+\infty} \frac{\gamma_t(a,\varepsilon)}{t} \tag{31}$$

we have $\beta < 1$ and so it follows from (30) that

$$M(Z_{m,u}, a, \alpha, \varepsilon) \le \varlimsup_{\tau\to+\infty} \sum_{p=\tau}^{+\infty} \beta^p = 0 \quad \text{and} \quad \alpha > P(a|_{Z_{m,u}}, \varepsilon). \tag{32}$$

Now take points $u_1, \ldots, u_r$ such that for each $u \in [\min b, \max b]$ there exists $j \in \{1, \ldots, r\}$ with $|u - u_j| < \delta$. Then

$$\mathcal{L}(Z) = \bigcup_{m\in\mathbb{N}} \bigcup_{i=1}^{r} Z_{m,u_i}$$

and so it follows from (31) and (32) together with the first property in Theorem 4 that

$$\begin{aligned} c + 5\delta + \varlimsup_{\varepsilon\to0} \varlimsup_{t\to+\infty} \frac{\gamma_t(a,\varepsilon)}{t} &\ge \varlimsup_{\varepsilon\to0} \sup_{m,i} P(a|_{Z_{m,u_i}}, \varepsilon) \\ &= \varlimsup_{\varepsilon\to0} P(a|_{\mathcal{L}(Z)}, \varepsilon) = P(a|_{\mathcal{L}(Z)}). \end{aligned}$$

Since $\delta$ is arbitrary and $a$ has tempered variation, we find that $P(a|_{\mathcal{L}(Z)}) \le c$.

*Step 3. Lower bound for the topological pressure.* In the remainder of the proof of the theorem we show that $P(a|_{\mathcal{L}(Z)}) \ge c$. First we establish an auxiliary result.

**Lemma 11.** *For each measure $\mu \in \mathcal{M}_Z$ there exists a $\Phi$-invariant function $\bar{b} \in L^1(X, \mu)$ such that*

$$\lim_{t\to+\infty} \frac{a_t}{t} = \lim_{t\to+\infty} \frac{1}{t} \int_0^t (b \circ \varphi_u)\,du = \bar{b}$$

*$\mu$-almost everywhere and in $L^1(X, \mu)$.*

*Proof of the lemma.* It follows from (29) that

$$\left|\frac{a_t(x)}{t}-\frac{1}{t}\int_0^t b(\varphi_u(x))\,du\right|\le\frac{1}{t}\sum_{k=1}^n\|b_{t-ks}\|_\infty+\frac{\|a_{t-ns}\|_\infty+(t-ns)\|b\|_\infty}{t}.$$

Let $n=\lfloor t/s\rfloor$. Then $t-ns\le s$ and since $\sup_{t\in[0,s]}\|a_t\|_\infty<+\infty$, we have

$$\sup_{t\ge 0}\big(\|a_{t-ns}\|_\infty+(t-ns)\|b\|_\infty\big)<+\infty.$$

Since $\sup_{t\in[0,T]}\|a_t\|_\infty<+\infty$ for all $T>0$, it follows from (27) that

$$\frac{1}{t}\left(a_t-\int_0^t(b\circ\varphi_u)\,du\right)\to 0$$

uniformly on $Z$ when $t\to+\infty$. On the other hand, since $b\in L^1(X,\mu)$, by Birkhoff's Ergodic theorem for flows there exists a $\Phi$-invariant function $\bar b\in L^1(X,\mu)$ such that

$$\lim_{t\to+\infty}\frac{1}{t}\int_0^t(b\circ\varphi_u)\,du=\bar b$$

$\mu$-almost everywhere and in $L^1(X,\mu)$. This yields the desired statement. □

Now we obtain a lower bound for the topological pressure.

**Lemma 12.** *For each ergodic measure $\mu\in\mathcal{M}_Z$, we have*

$$P(a|_Z)\ge h_\mu(\Phi)+\int_Z b\,d\mu.$$

*Proof of the lemma.* Given $\varepsilon>0$, there exist $\delta\in(0,\varepsilon)$, a measurable partition $\xi=\{C_1,\dots,C_m\}$ of $X$ and an open cover $\mathcal{V}=\{V_1,\dots,V_k\}$ of $X$ for some $k\ge m$ such that:

1. $\operatorname{diam}C_j\le\varepsilon$, $\overline{V_i}\subset C_i$ and $\mu(C_i\setminus V_i)<\delta^2$ for $i=1,\dots,m$;
2. the set $E=\bigcup_{i=m+1}^k V_i$ has measure $\mu(E)<\delta^2$.

Now we consider a measure $\nu$ in the ergodic decomposition of $\mu$ with respect to the time-1 map $\varphi_1$. The latter is described by a measure $\tau$ in the space $\mathcal{M}'$ of $\varphi_1$-invariant probability measures that is concentrated on the ergodic measures (with respect to $\varphi_1$). Note that $\nu(E)<\delta$ for $\nu$ in a set $\mathcal{M}_\delta\subset\mathcal{M}'$ of positive $\tau$-measure such that $\tau(\mathcal{M}_\delta)\to 1$ when $\delta\to 0$ since

$$\delta^2>\mu(E)=\int_{\mathcal{M}'}\nu(E)\,d\tau(\nu)\ge\int_{\mathcal{M}'\setminus\mathcal{M}_\delta}\nu(E)\,d\tau(\nu)\ge\delta\tau(\mathcal{M}'\setminus\mathcal{M}_\delta).$$

For each $x\in Z$ and $n\in\mathbb{N}$, let $t_n(x)$ be the number of integers $l\in[0,n)$ such that $\varphi_1^l(x)\in E$. By Birkhoff's ergodic theorem, since $\nu$ is ergodic for $\varphi_1$ we have

$$\lim_{n\to+\infty}\frac{t_n(x)}{n}=\lim_{n\to+\infty}\frac{1}{n}\sum_{j=0}^{n-1}\chi_E(\varphi_1^j(x))=\int_X\chi_E\,d\nu=\nu(E)\tag{33}$$

for $\nu$-almost every $x\in X$. On the other hand, by Lemma 11 and Birkhoff's ergodic theorem we have

$$\lim_{t\to+\infty}\frac{a_t(x)}{t}=\lim_{t\to+\infty}\frac{1}{t}\int_0^t(b\circ\varphi_u)(x)\,du=\int_Z b\,d\mu.\tag{34}$$

for $\mu$-almost every $x \in X$. By (33), (34) and Egorov's theorem, there exist $\nu \in \mathcal{M}_\delta$, $n_1 \in \mathbb{N}$ and a measurable set $A_1 \subset Z$ with $\nu(A_1) \geq 1-\delta$ such that

$$\frac{t_n(x)}{n} < 2\delta \quad \text{and} \quad \left|\frac{a_n(x)}{n} - \int_Z b\,d\mu\right| < \delta \tag{35}$$

for every $x \in A_1$ and $n > n_1$ (note that (34) holds for $\nu$-almost every $x \in X$, for $\tau$-almost every $\nu$, because it holds for $\mu$-almost every $x \in X$). Moreover, let

$$\xi_n = \bigvee_{j=0}^{n} \varphi_1^{-j}(\xi|_Z),$$

where $\xi|_Z$ is the partition induced by $\xi$ on $Z$. It follows from the Shannon–McMillan–Breiman theorem and Egorov's theorem that there exist $n_2 \in \mathbb{N}$ and a measurable set $A_2 \subset Z$ with $\nu(A_2) \geq 1-\delta$ such that

$$\nu(\xi_n(x)) \leq \exp\bigl[(-h_\nu(\varphi_1, \xi) + \delta)n\bigr] \tag{36}$$

for every $x \in A_2$ and $n > n_2$. Take

$$p = \max\{n_1, n_2\} \quad \text{and} \quad A = A_1 \cap A_2.$$

Note that $\nu(A) \geq 1-2\delta$. By construction, properties (35) and (36) hold for every $x \in A$ and $n > p$.

Now let $\Delta$ be a Lebesgue number of the cover $\mathcal{V}$ and take $\overline{\varepsilon} > 0$ such that $2\overline{\varepsilon} < \Delta$. Given $\alpha \in \mathbb{R}$, take $q \geq p$ such that for each $n \geq q$ there exists a set $\Gamma \subset X \times [n, +\infty)$ covering $Z$ with

$$\left|\sum_{(x,t)\in\Gamma} \exp(a(x,t,\overline{\varepsilon}) - \alpha t) - M(Z, a, \alpha, \overline{\varepsilon})\right| < \delta. \tag{37}$$

Given $l \in \mathbb{N}$, let

$$\Gamma_l = \bigl\{(x,l) \in \Gamma : B_l(x, \overline{\varepsilon}) \cap A \neq \varnothing\bigr\}$$

and define $B_l = \bigcup_{(x,t)\in\Gamma_l} B_t(x, \overline{\varepsilon})$. One can now proceed as in the proof of [13, Lemma 2] to show that

$$\operatorname{card}\Gamma_l \geq \nu(B_l \cap A)\exp\bigl[h_\nu(\varphi_1, \xi)l - (1 + 2\log\operatorname{card}\xi)l\delta\bigr] \tag{38}$$

for each $l \in \mathbb{N}$. Indeed, let $L_l$ be the number of elements $C$ of the partition $\xi_l$ such that $C \cap B_l \cap A \neq \varnothing$. It follows from (36) that

$$\nu(B_l \cap A) \leq \sum_{C\cap B_l\cap A\neq\varnothing} \nu(C) \leq L_l \exp\bigl[(-h_\nu(\varphi_1, \xi) + \delta)l\bigr]. \tag{39}$$

Note that by eventually making $\overline{\varepsilon}$ sufficiently small, for each $x \in Z$ there exist $i_1, \ldots, i_l \in \{1, \ldots, k\}$ such that $B_l(x, \overline{\varepsilon}) \subset V$, where

$$V = \bigcap_{j=1}^{l} \varphi_1^{-l+1} V_{i_j}$$

(this follows readily from the uniform continuity of the map $(t,x) \mapsto \varphi_t(x)$ on the compact set $[0,1] \times X$). Given $(x,l) \in \Gamma_l$, we have $B_l(x, \overline{\varepsilon}) \cap A_1 \neq \varnothing$ and so also $V \cap A_1 \neq \varnothing$. Hence, it follows from the first inequality in (35) that the number $S_{(x,l)}$ of elements $C$ of the partition $\xi_l$ such that $C\cap B_l(x,\overline{\varepsilon})\cap A\neq\varnothing$ satisfies

$$S_{(x,l)}\leq m^{2\delta l}=\exp(2\delta l\log m).$$

Therefore,

$$L_l\leq\sum_{(x,l)\in\Gamma_l}S_{(x,l)}\leq\operatorname{card}\Gamma_l\exp(2\delta l\log m). \tag{40}$$

Inequality (38) follows now readily from (39) and (40).

Observe that by the second inequality in (35) we have

$$\sup_{B_l(x,\overline{\varepsilon})}a_l\geq l\left(\int_Z b\,d\mu-\delta\right)-\gamma_l(a,\overline{\varepsilon})$$

for all $l\geq q$ and $(x,t)\in\Gamma_l$. Therefore,

$$\begin{aligned}&\sum_{(x,t)\in\Gamma}\exp(a(x,t,\overline{\varepsilon})-\alpha t)\\&\geq\sum_{l=q}^{+\infty}\sum_{(x,t)\in\Gamma_l}\exp\biggl(\sup_{B_l(x,\overline{\varepsilon})}a_l-\alpha l\biggr)\\&\geq\sum_{l=q}^{+\infty}\operatorname{card}\Gamma_l\exp\left[\left(-\alpha+\int_Z b\,d\mu-\delta\right)l-\gamma_l(a,\overline{\varepsilon})\right]\\&\geq\sum_{l=q}^{+\infty}\nu(B_l\cap A)\\&\quad\times\exp\left[\left(h_\nu(\varphi_1,\xi)+\int_Z b\,d\mu-\frac{\gamma_l(a,\overline{\varepsilon})}{l}-\alpha\right)l-2(1+\log\operatorname{card}\xi)l\delta\right].\end{aligned}$$

Without loss of generality one can assume that $q$ is sufficiently large so that

$$\frac{\gamma_l(a,\overline{\varepsilon})}{l}\leq\varlimsup_{t\to+\infty}\frac{\gamma_t(a,\overline{\varepsilon})}{t}+\delta$$

for all $l\geq q$. Now take

$$\alpha<h_\nu(\varphi_1,\xi)+\int_Z b\,d\mu-\varlimsup_{t\to+\infty}\frac{\gamma_t(a,\overline{\varepsilon})}{t}.$$

Without loss of generality one can also assume that $\delta$ is so small such that

$$\alpha<h_\nu(\varphi_1,\xi)+\int_Z b\,d\mu-\varlimsup_{t\to+\infty}\frac{\gamma_t(a,\overline{\varepsilon})}{t}-2(1+\log\operatorname{card}\xi)\delta-\delta.$$

Then

$$\sum_{(x,t)\in\Gamma}\exp(a(x,t,\overline{\varepsilon})-\alpha t)\geq\sum_{l=q}^{+\infty}\nu(B_l\cap A)\geq\nu(A)\geq 1-2\delta$$

and so it follows from (37) that

$$M(Z,a,\alpha,\overline{\varepsilon})>1-3\delta>0.$$

Therefore, $P(a|_Z,\overline{\varepsilon})\geq\alpha$, which implies that

$$P(a|_Z,\overline{\varepsilon})\geq h_\nu(\varphi_1,\xi)+\int_Z b\,d\mu-\varlimsup_{t\to+\infty}\frac{\gamma_t(a,\overline{\varepsilon})}{t}.$$

Finally, we consider measurable partitions $\xi_l$ and open covers $\mathcal{V}_l$ as before with $\varepsilon = 1/l$. For each $l$ take $\overline{\varepsilon}_l > 0$ such that $2\overline{\varepsilon}_l < 1/l$ is a Lebesgue number of the cover $\mathcal{V}_l$. Since $\operatorname{diam} \xi_l \to 0$ when $l \to +\infty$, it follows that

$$\lim_{l\to+\infty} h_\nu(\varphi_1, \xi_l) = h_\nu(\varphi_1).$$

Moreover, since the family $a$ has tempered variation property, we obtain

$$\begin{aligned} P(a|_Z) &= \lim_{l\to+\infty} P(a|_Z, \overline{\varepsilon}_l) \\ &\geq \lim_{l\to+\infty} h_\nu(\varphi_1, \xi_l) + \int_Z b\, d\mu - \lim_{l\to+\infty} \overline{\lim_{t\to+\infty}} \frac{\gamma_t(a, \overline{\varepsilon}_l)}{t} \\ &= h_\nu(\varphi_1) + \int_Z b\, d\mu. \end{aligned}$$

Integrating with respect to $\nu$ gives

$$P(a|_Z) \geq \int_{\mathcal{M}_\delta} h_\nu(\varphi_1)\, d\tau(\nu) + \int_Z b\, d\mu$$

and letting $\delta \to 0$ yields the inequality

$$\begin{aligned} P(a|_Z) &\geq \int_{\mathcal{M}'} h_\nu(\varphi_1)\, d\tau(\nu) + \int_Z b\, d\mu \\ &= h_\mu(\varphi_1) + \int_Z b\, d\mu = h_\mu(\Phi) + \int_Z b\, d\mu. \end{aligned}$$

This completes the proof of the lemma. □

When $\mu \in \mathcal{M}_Z$ is ergodic, $Z_\mu$ is a nonempty $\Phi$-invariant subset of $\mathcal{L}(Z)$ with $\mu(Z_\mu) = 1$. Hence, it follows from Lemma 12 that

$$P(a|_{\mathcal{L}(Z)}) \geq P(a|_{Z_\mu}) \geq h_\mu(\Phi) + \int_{Z_\mu} b\, d\mu = h_\mu(\Phi) + \int_Z b\, d\mu.$$

When $\mu \in \mathcal{M}_Z$ is arbitrary, one can decompose $X$ into ergodic components and the previous argument shows that

$$P(a|_{\mathcal{L}(Z)}) \geq \sup_{\mu\in\mathcal{M}_Z} \left( h_\mu(\Phi) + \int_Z b\, d\mu \right).$$

This completes the proof of the theorem. □

It follows from Theorem 9 that if $V(x) \cap \mathcal{M}_Z \neq \varnothing$ for each $x \in Z$, and so in particular if $Z$ is compact and $\Phi$-invariant, then

$$P(a|_Z) = \sup_{\mu\in\mathcal{M}_Z} \left( h_\mu(\Phi) + \int_Z b\, d\mu \right).$$

## References


[1] L. Barreira, *A non-additive thermodynamic formalism and applications to dimension theory of hyperbolic dynamical systems*, Ergodic Theory Dynam. Systems **16** (1996), 871–927.

[2] L. Barreira, *Dimension and Recurrence in Hyperbolic Dynamics*, Progress in Mathematics 272, Birkhäuser, 2008.

[3] L. Barreira, *Thermodynamic Formalism and Applications to Dimension Theory*, Progress in Mathematics 294, Birkhuser, 2011.

[4] R. Bowen, *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms*, Springer Lectures Notes in Mathematics 470, Springer Verlag, 1975.

[5] R. Bowen, *Hausdorff dimension of quasi-circles*, Inst. Hautes Études Sci. Publ. Math. **50** (1979), 259–273.
[6] A. Douady and J. Oesterlé, *Dimension de Hausdorff des attracteurs*, C. R. Acad. Sc. Paris **290** (1980), 1135–1138.
[7] K. Falconer, *A subadditive thermodynamic formalism for mixing repellers*, J. Phys. A **21** (1988), L737–L742.
[8] G. Keller, *Equilibrium States in Ergodic Theory*, London Mathematical Society Student Texts 42, Cambridge University Press, 1998.
[9] H. McCluskey and A. Manning, *Hausdorff dimension for horseshoes*, Ergodic Theory Dynam. Systems **3** (1983), 251–260.
[10] W. Parry and M. Pollicott, *Zeta Functions and the Periodic Orbit Structure of Hyperbolic Dynamics*, Astérisque 187-188, 1990.
[11] Ya. Pesin, *Dimension type characteristics for invariant sets of dynamical systems*, Russian Math. Surveys **43** (1988), 111–151.
[12] Ya. Pesin, *Dimension Theory in Dynamical Systems. Contemporary Views and Applications*, Chicago Lectures in Mathematics, Chicago University Press, 1997.
[13] Ya. Pesin and B. Pitskel', *Topological pressure and the variational principle for noncompact sets*, Functional Anal. Appl. **18** (1984), 307–318.
[14] D. Ruelle, *Statistical mechanics on a compact set with* $\mathbb{Z}^\nu$ *action satisfying expansiveness and specification*, Trans. Amer. Math. Soc. **185** (1973), 237–251.
[15] D. Ruelle, *Thermodynamic Formalism*, Encyclopedia of mathematics and its applications 5, Addison-Wesley, 1978.
[16] D. Ruelle, *Repellers for real analytic maps*, Ergodic Theory Dynam. Systems **2** (1982), 99–107.
[17] P. Walters, *A variational principle for the pressure of continuous transformations*, Amer. J. Math. **97** (1976), 937–971.

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