NONLINEAR THERMODYNAMIC FORMALISM FOR FLOWS

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Abstract. We introduce a version of the nonlinear thermodynamic formalism for flows. Moreover, we discuss the existence, uniqueness, and characterization of equilibrium measures for almost additive families of continuous functions with tempered variation. We also consider with some care the special case of additive families for which it is possible to strengthen some of the results. The proofs are mainly based on multifractal analysis.

1. Introduction

Our main aim is introduce a version for flows of the *nonlinear thermodynamic formalism* recently introduced in [14] for a dynamics with discrete time. Besides introducing the notion of nonlinear topological pressure of a family of continuous functions with respect to a flow, we discuss the existence, uniqueness, and characterization of equilibrium measures for an almost additive family of continuous functions with tempered variation with respect to a flow. We consider in particular the special case of an additive family of continuous functions for which is it is possible to strengthen some of the results.

The topological pressure $P(\phi)$ of a continuous function $\phi\colon X \to \mathbb{R}$ with respect to a continuous map $T\colon X \to X$ on a compact metric space was introduced by Ruelle [26] for expansive maps and by Walters [28] in the general case. They also established the variational principle

$$P(\phi) = \sup_{\mu} \left(h_\mu(T) + \int_X \phi \, d\mu \right),$$

with the supremum taken over all T-invariant probability measures μ on X and where $h_\mu(T)$ denotes the Kolmogorov–Sinai entropy with respect to μ. For corresponding results for continuous time we refer the reader to [12, 22]. The theory is quite broad and has many applications. We refer the reader to the books [1, 11, 19, 20, 22, 23, 27, 29] for details and further references.

Given a continuous function $F\colon \mathbb{R} \to \mathbb{R}$, the *nonlinear topological pressure* $P_F(\phi)$ of a continuous function $\phi\colon X \to \mathbb{R}$ essentially replaces in the classical notion the Birkhoff sum

$$(S_n\phi)(x) = \sum_{k=0}^{n-1} \phi(T^k(x))$$

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by the expression
$$nF\left(\frac{(S_n\phi)(x)}{n}\right).$$
The corresponding formalism introduced in [14] also includes a variational principle for the nonlinear topological pressure. Namely, under an additional assumption of abundance of ergodic measures (see Section 3 for the definition), they proved that
$$P_F(\phi) = \sup_\mu \left(h_\mu(T) + F\left(\int_X \phi\,d\mu\right)\right), \tag{1}$$
with the supremum taken over all T-invariant probability measures μ on X. They also characterized the equilibrium measures, that is, the invariant probability measures at which the supremum in (1) is attained.

Now we describe briefly our main results. Let $\Phi = (\phi_t)_{t\in\mathbb{R}}$ be a continuous flow on a compact metric space X and let $F\colon \mathbb{R}^d \to \mathbb{R}$ be a continuous function for some $d \in \mathbb{N}$. We start by introducing the notion of the *nonlinear topological pressure* $P_F(A|_Z)$ of a collection $A = (a^1, \dots, a^d)$ of families $a^i = (a^i_t)_{t\ge 0}$ of continuous functions $a^i_t\colon X \to \mathbb{R}$ on a set $Z \subset X$ as a Carathéodory dimension characteristic (see Section 2 for the definition). This allows us to consider arbitrary sets Z, and so possibly noncompact and noninvariant sets. We say that a family of functions $a = (a_t)_{t\ge0}$ is *almost additive* (with respect to Φ) if there exists $C > 0$ such that
$$-C \le a_{t+s} - a_t - a_s \circ \phi_t \le C$$
for every $t, s > 0$. Now assume that each family a^i in A is almost additive, has tempered variation, and satisfies $\sup_{t\in[0,s]} \|a^i_t\|_\infty < \infty$ for some $s > 0$. Assuming that the pair (Φ, A) has an abundance of ergodic measures (see Section 3), we establish the variational principle for the nonlinear topological pressure
$$P_F(A) = \sup_{\mu\in\mathcal{M}} \left\{h_\mu(\Phi) + F\left(\lim_{t\to\infty}\frac{1}{t}\int_X A_t\,d\mu\right)\right\},$$
where $\mathcal{M}$ denotes the set of all Φ-invariant probability measures on X and $A_t = (a^1_t, \dots, a^d_t)$ (see Theorem 4).

In Section 4 we discuss the existence and characterization of equilibrium measures using in particular some machinery coming from multifractal analysis. Here we formulate only our main result. In order to do that we first need to introduce a few additional notions. Let
$$L(A) = \left\{\lim_{t\to\infty}\frac{1}{t}\int_X A_t\,d\mu : \mu \in \mathcal{M}\right\}$$
and
$$\mathcal{M}(z) = \left\{\mu \in \mathcal{M} : \lim_{t\to\infty}\frac{1}{t}\int_X A_t\,d\mu = z\right\}.$$
We also define a function $h\colon L(A) \to \mathbb{R}$ by
$$h(z) = \sup\{h_\mu(\Phi) : \mu \in \mathcal{M}(z)\}.$$
We say that a pair (Φ, A) is C^1*-regular* if the following properties hold:

1. each family in $\operatorname{span}\{a^1, \dots, a^d, 1\}$ has a unique equilibrium measure for the nonadditive topological pressure and $\operatorname{int} L(A) \ne \emptyset$;

2. the map $\mu \mapsto h_\mu(\Phi)$ is upper-semicontinuous.

Examples can be obtained from locally maximal hyperbolic sets. Finally, given a continuous function $F\colon \mathbb{R}^d \to \mathbb{R}$, we consider the set

$$K(F,A) = \left\{ \lim_{t\to\infty} \frac{1}{t} \int_X A_t \, d\mu : \mu \text{ is an equilibrium measure for } (F,A) \right\}.$$

The following theorem is our main result (see Theorem 6). In particular it gives a characterization of equilibrium measures.

Theorem 1. *If the pair (Φ, A) is C^1-regular, then for each continuous function $F\colon \mathbb{R}^d \to \mathbb{R}$ the following properties hold:*

1. *$K(F,A)$ is a nonempty compact set;*
2. *$K(F,A)$ is the set of maximizers of the function $h+F$;*
3. *if $K(F,A) \subset \operatorname{int} L(A)$, then the equilibrium measures for (F,A) are the elements of the set $\{\nu_z : z \in K(F,A)\}$, where ν_z is an ergodic measure that is the unique equilibrium measure for some almost additive family.*

Our proof of Theorem 1 depends substantially on the multifractal analysis developed in [4], which thus encounters here a welcome nontrivial application. More precisely, we use the following simplified multidimensional version of Theorem 8 in that paper. Consider the level sets

$$C_z(A) = \left\{ x \in X : \lim_{t\to\infty} \frac{A_t(x)}{t} = z \right\},$$

and let $\mathcal{E}(X)$ be the set of almost additive families (with respect to Φ) with a unique equilibrium measure.

Theorem 2. *Let Φ be a continuous flow on a compact metric space X such that the map $\mu \mapsto h_\mu(\Phi)$ is upper-semicontinuous and assume that $\operatorname{span}\{a^{(1)}, \ldots, a^d, 1\} \subset \mathcal{E}(X)$. If $z \in \operatorname{int} L(A)$, then $C_z(A) \neq \emptyset$ and the following properties hold:*

1. $$h_{\text{top}}(\Phi|C_z(A)) = \max\left\{ h_\mu(\Phi) : \mu \in \mathcal{M} \text{ and } \lim_{t\to\infty} \frac{1}{t} \int_X A_t \, d\mu = z \right\};$$
2. *there exists an ergodic measure $\nu_z \in \mathcal{M}$ such that*
$$\lim_{t\to\infty} \frac{1}{t} \int_X A_t d\nu_z = z, \ \nu_z(C_z(A)) = 1 \text{ and } h_{\nu_z}(\Phi) = h_{\text{top}}(\Phi|C_z(A));$$
3. *the function $z \mapsto h_{\text{top}}(\Phi|C_z(A))$ is continuous on $\operatorname{int} L(A)$.*

Related results for a dynamics with discrete time were obtained independently in [6] and [14]. While [14] uses mainly convex analysis, our approach in the present paper is more inspired in [6], which takes into account some connections between the nonlinear thermodynamic formalism and multifractal analysis (for a dynamics with discrete time). In this sense, the present work also shows that there is a quite interesting connection between the nonlinear thermodynamic formalism and multifractal analysis for a dynamics with continuous time, which in fact is another main aspect of our work.

2. Nonlinear topological pressure for flows

Let $\Phi = (\phi_t)_{t\in\mathbb{R}}$ be a continuous flow on a compact metric space X. For each $t > 0$, we consider the distance d_t on X given by

$$d_t(x, y) = \max\bigl\{d(\phi_s(x), \phi_s(y)) : s \in [0, t]\bigr\}$$

and for each $x \in X$ and $\varepsilon > 0$ we define

$$B_t(x, \varepsilon) = \{y \in X : d_t(y, x) < \varepsilon\}.$$

A family $a = (a_t)_{t\geq 0}$ of continuous functions $a_t \colon X \to \mathbb{R}$ is said to have *tempered variation* if

$$\lim_{\varepsilon\to 0} \varlimsup_{t\to\infty} \frac{\gamma_t(a, \varepsilon)}{t} = 0, \tag{2}$$

where

$$\gamma_t(a, \varepsilon) = \sup\bigl\{|a_t(y) - a_t(z)| : y, z \in B_t(x, \varepsilon) \text{ for some } x \in X\bigr\}.$$

Now let $A = (a^1, \ldots, a^d)$ be a finite collection of almost additive families of continuous functions with tempered variation and consider a continuous function $F \colon \mathbb{R}^d \to \mathbb{R}$. We note that

$$\lim_{\varepsilon\to 0} \varlimsup_{t\to\infty} \frac{\bar\gamma_t(A, \varepsilon)}{t} = 0, \tag{3}$$

where $\bar\gamma_t(A, \varepsilon)$ is given by

$$\sup\biggl\{t\biggl|F\biggl(\frac{A_t(y)}{t}\biggr) - F\biggl(\frac{A_t(z)}{t}\biggr)\biggr| : y, z \in B_t(x, \varepsilon) \text{ for some } x \in X\biggr\}$$

with $A_t(x) = (a^1_t(x), \ldots, a^d_t(x))$ for any $x \in X$ and $t \geq 0$. This can be shown as follows. Since each family in A is almost additive, for each $i \in \{1, \ldots, d\}$ there exists a constant $K_i > 0$ such that

$$\frac{|a^i_t(x)|}{t} \leq K_i \quad \text{for } t > 0 \text{ and } x \in X$$

(see [5]). Hence, we only need to consider F restricted to the compact set $[-K_1, K_1] \times \cdots \times [-K_d, K_d] \subset \mathbb{R}^d$. By the uniform continuity of F on this set, for each $\kappa > 0$ there exists $\eta > 0$ such that

$$\biggl|F\biggl(\frac{A_t(y)}{t}\biggr) - F\biggl(\frac{A_t(z)}{t}\biggr)\biggr| < \kappa \quad \text{whenever } \biggl\|\frac{A_t(y)}{t} - \frac{A_t(z)}{t}\biggr\| < \eta, \tag{4}$$

where $\|\cdot\|$ denotes the l^∞ norm on $\mathbb{R}^d$. Let

$$\gamma^i(\varepsilon) = \varlimsup_{t\to\infty} \frac{\gamma_t(a^i, \varepsilon)}{t} \quad \text{for each } i \in \{1, \ldots, d\}.$$

There exists $\tau > 0$ such that

$$\gamma_t(a^i, \varepsilon)/t < \gamma^i(\varepsilon) + \eta/2 \quad \text{for } t > \tau \text{ and } i \in \{1, \ldots, d\}.$$

By property (2), for any sufficiently small $\varepsilon > 0$ we have $\gamma^i(\varepsilon) < \eta/2$ and so $\gamma_t(a^i, \varepsilon)/t < \eta$ for $i \in \{1, \ldots, d\}$. It follows from (4) that $\bar\gamma_t(A, \varepsilon)/t < \kappa$ for $t > \tau$. Property (3) follows now from the arbitrariness of κ.

Following [3], we introduce a nonlinear version of the nonadditive topological pressure for noncompact sets. Given $\varepsilon > 0$, we say that a set $\Gamma \subset X \times \mathbb{R}_0^+$ *covers* $Z \subset X$ if

$$\bigcup_{(x,t)\in\Gamma} B_t(x,\varepsilon) \supset Z$$

and we write

$$A_F(x,t,\varepsilon) = \sup\left\{ tF\left(\frac{A_t(y)}{t}\right) : y \in B_t(x,\varepsilon)\right\} \quad \text{for } (x,t) \in \Gamma.$$

For each $Z \subset X$ and $\alpha \in \mathbb{R}$, let

$$M_F(Z,A,\alpha,\varepsilon) = \lim_{T\to+\infty} \inf_{\Gamma} \sum_{(x,t)\in\Gamma} \exp(A_F(x,t,\varepsilon) - \alpha t) \tag{5}$$

with the infimum taken over all countable sets $\Gamma \subset X \times [T,+\infty)$ covering Z. When α goes from $-\infty$ to $+\infty$, the quantity in (5) jumps from $+\infty$ to 0 at a unique value and we define

$$P_F(A|_Z,\varepsilon) = \inf\big\{\alpha \in \mathbb{R} : M_F(Z,A,\alpha,\varepsilon) = 0\big\}.$$

Following the proof of Theorem 1 in [3], one can show that the limit

$$P_F(A|_Z) = \lim_{\varepsilon\to 0} P_F(A|_Z,\varepsilon)$$

exists. It is called the *nonlinear topological pressure* of $A = (a^1,\dots,a^d)$ on the set Z. When $d = 1$ and F is the identity map, we recover the nonadditive topological pressure in [3].

Proposition 3. *The following properties hold:*

1. *if $Z_1 \subset Z_2$, then $P_F(A|_{Z_1}) \le P_F(A|_{Z_2})$;*
2. *if $Z = \bigcup_{i\in I} Z_i$ with I countable, then $P_F(A|_Z) = \sup_{i\in I} P_F(A|_{Z_i})$.*

Proof. For the first property, observe that for each $\varepsilon > 0$ and $\alpha \in \mathbb{R}$ we have

$$M_F(Z_1,A,\alpha,\varepsilon) \le M_F(Z_2,A,\alpha,\varepsilon).$$

Then,

$$P_F(A|_{Z_1}) = \lim_{\varepsilon\to 0} P_F(A|_{Z_1},\varepsilon) \le \lim_{\varepsilon\to 0} P_F(A|_{Z_2},\varepsilon) = P_F(A|_{Z_2}).$$

Now we establish the second property. Since $Z_i \subset Z$ for every $i \in I$, the first property says that $P_F(A|_{Z_i}) \le P_F(A|_Z)$ for $i \in I$, which implies that

$$\sup_{i\in I} P_F(A|_{Z_i}) \le P_F(A|_Z).$$

To prove the reverse inequality, take $\varepsilon > 0$ and $\alpha > \sup_{i\in I} P_F(A|_{Z_i},\varepsilon)$. Then $M_F(Z_i,A,\alpha,\varepsilon) = 0$ for $i \in I$. By definition, given $\eta > 0$ and $T > 0$, for each $i \in I$ there exists $\Gamma_i \subset X \times [T,\infty)$ covering Z_i such that

$$\sum_{(x,t)\in\Gamma_i} \exp(A_F(x,t,\varepsilon) - \alpha t) < \frac{\eta}{2^i}.$$

Hence, $\Gamma = \bigcup_{i\in I} \Gamma_i$ covers Z and

$$\sum_{(x,t)\in\Gamma} \exp(A_F(x,t,\varepsilon) - \alpha t) \le \sum_{i\in I} \sum_{(x,t)\in\Gamma_i} \exp(A_F(x,t,\varepsilon) - \alpha t) \le \sum_{i\in I} \frac{\eta}{2^i} \le \eta.$$

Therefore, $M_F(Z,A,\alpha,\varepsilon) \le \eta$ and since the constant η is arbitrary, we have $M_F(Z,A,\alpha,\varepsilon) = 0$. Hence, by definition $\alpha \ge P_F(A|_Z,\varepsilon)$. Finally, letting $\alpha \to \sup_{i\in I} P_F(A|_{Z_i})$, we obtain

$$\sup_{i\in I} P_F(A|_{Z_i},\varepsilon) \ge P_F(A|_Z,\varepsilon). \tag{6}$$

Proceeding as in the proof of Theorem 4 in [3], one can show that

$$P_F(A|_{Z_i}) \ge P_F(A|_{Z_i},\varepsilon) - \varlimsup_{t\to\infty} \frac{\bar\gamma_t(A,\varepsilon)}{t}. \tag{7}$$

Hence, it follows from (6) and (7) that

$$\begin{aligned}\sup_{i\in I} P_F(A|_{Z_i}) &\ge \sup_{i\in I} P_F(A|_{Z_i},\varepsilon) - \varlimsup_{t\to\infty} \frac{\bar\gamma_t(A,\varepsilon)}{t}\\ &\ge P_F(A|_Z,\varepsilon) - \varlimsup_{t\to\infty} \frac{\bar\gamma_t(A,\varepsilon)}{t}.\end{aligned}$$

Letting $\varepsilon \to 0$ and using property (3), we obtain

$$\sup_{i\in I} P_F(A|_{Z_i}) \ge P_F(A|_Z),$$

which completes the proof of the proposition. □

3. Variational principle

Let Φ be a continuous flow on a compact metric space X. In this section, we consider the case when each family in the collection $A = (a^1,\dots,a^d)$ belongs to the set $\mathcal{A}(X)$ of almost additive families of continuous functions with tempered variation such that $\sup_{t\in[0,s]} \|a_t\|_\infty < \infty$ for some $s > 0$. When this happens, we shall simply write $A \in \mathcal{A}(X)^d$. In this case, for each $\mu \in \mathcal{M}$ in the set of Φ-invariant probability measures on X we have

$$\lim_{t\to\infty} \frac{1}{t}\int_X a_t^i\,d\mu = \int_X \lim_{t\to\infty} \frac{a_t^i(x)}{t}\,d\mu(x)$$

for each $i \in \{1,\dots,d\}$ (see [5] for details). Following [14], we say that the pair (Φ, A) has *an abundance of ergodic measures* if for each $\mu \in \mathcal{M}$ and $\varepsilon > 0$, there exists an ergodic measure $\nu \in \mathcal{M}$ such that

$$h_\nu(\Phi) + F\left(\lim_{t\to\infty} \frac{1}{t}\int_X A_t d\nu\right) \ge h_\mu(\Phi) + F\left(\lim_{t\to\infty} \frac{1}{t}\int_X A_t d\mu\right) - \varepsilon.$$

When $F\colon \mathbb{R}^d \to \mathbb{R}$ is convex, the pair (Φ, A) has an abundance of ergodic measures for every continuous flow Φ and every collection A of almost additive families of continuous functions. In fact, by Kingman's subadditive ergodic theorem, for each $\mu \in \mathcal{M}$ and each $i \in \{1,\dots,d\}$ the limit

$$b^i(x) := \lim_{t\to\infty} \frac{a_t^i(x)}{t}$$

exists for μ-almost every $x \in X$ and b^i is a measurable function. Now let $\mu \in \mathcal{M}$ be an arbitrary measure and consider its ergodic decomposition with respect to Φ (described by a probability measure θ on $\mathcal{M}$ concentrated on the subset of ergodic measures $\mathcal{M}_{\text{erg}}$). Using Jensen's inequality, it follows from the continuity of F that

$$\begin{aligned}\lim_{t\to\infty} F\left(\frac{1}{t}\int_X A_t\,d\mu\right) &= F\left(\int_{\mathcal{M}}\left(\int_X b^1\,d\nu\right)d\theta(\nu),\dots,\int_{\mathcal{M}}\left(\int_X b^d\,d\nu\right)d\theta(\nu)\right)\\ &\le \int_{\mathcal{M}} F\left(\int_X b^1\,d\nu,\dots,\int_X b^d\,d\nu\right)d\theta(\nu)\\ &= \int_{\mathcal{M}}\lim_{t\to\infty} F\left(\frac{1}{t}\int_X A_t\,d\nu\right)d\theta(\nu).\end{aligned}$$

Since $h_\mu(\Phi) = \int_{\mathcal{M}} h_\nu(\Phi)d\theta(\nu)$ (see for example Theorem 9.6.2 in [21]) we obtain

$$h_\mu(\Phi) + \lim_{t\to\infty} F\left(\frac{1}{t}\int_X A_t\,d\mu\right) \le \int_{\mathcal{M}}\left[h_\nu(\Phi) + \lim_{t\to\infty} F\left(\frac{1}{t}\int_X A_t\,d\nu\right)\right]d\theta(\nu),$$

which, in particular, guarantees the existence of an abundance of ergodic measures for (Φ, A).

Moreover, we say that Φ has *entropy density of ergodic measures* if for every $\mu \in \mathcal{M}$ there exist ergodic invariant measures ν_n for $n \in \mathbb{N}$ such that $\nu_n \to \mu$ in the weak* topology and $h_{\nu_n}(\Phi) \to h_\mu(\Phi)$ when $n \to \infty$. Note that if Φ has entropy density of ergodic measures, then (Φ, A) has an abundance of ergodic measures for every collection A of almost additive families of continuous functions.

We say that a continuous flow Φ has *weak specification at scale* $\delta > 0$ if there exists $\gamma > 0$ such that for each finite set of orbit segments $\{(x_i, t_i)\}_{i=1}^k$ there exists a point $y \in X$ and times $\gamma_1, \dots, \gamma_{k-1} \in [0, \gamma]$ such that

$$d_{t_j}(\phi_{s_{j-1}+\gamma_{j-1}}(y), x_j) < \delta \quad \text{for } j = 1, \dots, k,$$

where

$$s_j = \sum_{i=1}^{j} t_i + \sum_{i=1}^{j-1}\gamma_i \quad \text{and} \quad s_0 = \gamma_0 = 0.$$

The flow Φ is said to have *weak specification* if it has weak specification at every scale. When for every $\delta > 0$ one can take y to be periodic and the times γ_i to be close to γ, we say that Φ has *specification* (see for example [10]). It was proved in [16] that every expansive flow with the weak specification property has entropy density of ergodic measures. This includes locally maximal hyperbolic sets for topologically mixing C^1 flows.

Theorem 4. *Let Φ be a continuous flow on a compact metric space X and take $A \in \mathcal{A}(X)^d$. Given a continuous function $F\colon \mathbb{R}^d \to \mathbb{R}$, if the pair (Φ, A) has an abundance of ergodic measures, then*

$$\begin{aligned}P_F(A) &= \sup_{\mu\in\mathcal{M}}\left\{h_\mu(\Phi) + F\left(\lim_{t\to\infty}\frac{1}{t}\int_X A_t\,d\mu\right)\right\}\\ &= \sup_{\mu\in\mathcal{M}}\left\{h_\mu(\Phi) + \lim_{t\to\infty} F\left(\frac{1}{t}\int_X A_t\,d\mu\right)\right\}.\end{aligned} \tag{8}$$

Proof. We divide the proof into steps.

Step 1. First we obtain the inequality

$$P_F(A) \le \sup_{\mu \in \mathcal{M}} \left\{ h_\mu(\Phi) + \lim_{t\to\infty} F\left(\frac{1}{t}\int_X A_t\, d\mu\right)\right\}. \tag{9}$$

Given $x \in X$, we define a probability measure on X by

$$\mu_{x,t} = \frac{1}{t}\int_0^t \delta_{\phi_s(x)}\, ds,$$

where δ_y is the probability measure concentrated on y. Let also $V(x)$ be the set of all sublimits of the family $(\mu_{x,t})_{t>0}$.

Lemma 1 ([5, Lemma 2.2])**.** *Let $a \in \mathcal{A}(X)$. Given $x \in X$ and $\mu \in V(x)$, there exists an increasing sequence $(t_n)_{n\in\mathbb{N}}$ such that*

$$\lim_{n\to\infty} \frac{a_{t_n}(x)}{t_n} = \lim_{t\to\infty} \frac{1}{t}\int_X a_t\, d\mu.$$

Lemma 2. *Let $\Gamma \subset X \times \{1\}$ be a finite cover of X. For the open cover $\mathcal{V} = \{V_1, \ldots, V_r\}$ of X, where $V_j = B_1(x_j, \varepsilon/2)$ with $(x_j, 1) \in \Gamma$, there exist $m, T \in \mathbb{N}$ with T arbitrary large and a sequence $U = V_{i_1} \cdots V_{i_T}$ such that:*

1. *$x \in \bigcap_{r=1}^T \phi_{-r+1} V_{i_r}$ and*
$$F\left(\frac{A_T(x)}{T}\right) \le \lim_{t\to\infty} F\left(\frac{1}{t}\int_X A_t\, d\mu\right) + \delta;$$
2. *there exists a subset $V \in (\mathcal{V}^m)^k$ of U of length $km \ge T - m$ such that $H(V) \le m(h_\mu(\Phi) + \delta)$.*

Proof of the lemma. Since $F\colon \mathbb{R}^d \to \mathbb{R}$ is uniformly continuous on the compact set $S = [r_1, s_1] \times \cdots \times [r_d, s_d]$ with

$$r_i = \inf_{\mu\in\mathcal{M}} \lim_{t\to\infty} \frac{1}{t}\int_X a_t^i\, d\mu \quad \text{and} \quad s_i = \sup_{\mu\in\mathcal{M}} \lim_{t\to\infty} \frac{1}{t}\int_X a_t^i\, d\mu,$$

given $\delta > 0$, there exists $\rho > 0$ such that $|F(x) - F(y)| < \delta$ whenever $x, y \in S$ and $\|x - y\| < \rho$. By Lemma 1, for each $i \in \{1, \ldots, d\}$ there exists an increasing sequence $(t_n)_{n\in\mathbb{N}}$ (possibly depending on the family a^i) such that

$$\left| \frac{a_{t_n}^i(x)}{t_n} - \lim_{t\to\infty} \frac{1}{t}\int_X a_t^i\, d\mu \right| < \rho$$

for any sufficiently large $n \in \mathbb{N}$. Then, taking T sufficiently large, we obtain

$$F\left(\frac{A_T(x)}{T}\right) \le \lim_{t\to\infty} F\left(\frac{1}{t}\int_X A_t\, d\mu\right) + \delta,$$

which proves the first property. The second property follows directly from Lemma 2.3 in [5]. $\square$

Given $\delta > 0$, $m \in \mathbb{N}$ and $u \in \mathbb{R}$, let $X_{m,u}$ be the set of all points $x \in X$ satisfying the two properties in Lemma 2 for some measure $\mu \in V(x)$ with

$$u - \delta \le \lim_{t\to\infty} F\left(\frac{1}{t}\int_X A_t\, d\mu\right) \le u + \delta.$$

Moreover, let n_T be the number of sequences $U \in \mathcal{V}^T$ satisfying the two properties in Lemma 2 for some point $x \in X_{m,u}$. Proceeding as in the proof of Lemma 4.3.3 in [1], we find that

$$n_T \le \exp[T(h_\mu(\Phi) + 2\delta)]$$

for any sufficiently large T. For each $\tau \in \mathbb{N}$, the collection of all $U \in \mathcal{V}^T$ satisfying the two properties in Lemma 2 for some $x \in X_{m,u}$ and $T > \tau$ covers $X_{m,u}$. Then

$$\begin{aligned}
&M_F(X_{m,u}, A, \alpha, \varepsilon) \\
&\le \varlimsup_{\tau\to\infty} \sum_{T=\tau}^{+\infty} n_T \exp\left[\bar{\gamma}_T(A,\varepsilon) + T\left(\lim_{t\to\infty} F\left(\frac{1}{t}\int_X A_t\, d\mu\right) + \delta\right) - \alpha T\right] \\
&\le \varlimsup_{\tau\to\infty} \sum_{T=\tau}^{+\infty} \exp\left[T\left(h_\mu(\Phi) + \bar{\gamma}(\varepsilon) + \lim_{t\to\infty} F\left(\frac{1}{t}\int_X A_t\, d\mu\right) - \alpha + 3\delta\right)\right],
\end{aligned}$$

where

$$\bar{\gamma}(\varepsilon) = \varlimsup_{t\to\infty} \frac{\bar{\gamma}_t(A,\varepsilon)}{t}.$$

Now take

$$K = \sup_{\mu\in\mathcal{M}} \left\{ h_\mu(\Phi) + \lim_{t\to\infty} F\left(\frac{1}{t}\int_X A_t\, d\mu\right)\right\}$$

and

$$\theta = \exp\left[K - \alpha + 3\delta + \bar{\gamma}(\varepsilon)\right].$$

For $\alpha > K + 3\delta + \bar{\gamma}(\varepsilon)$ we have $\theta < 1$, and so

$$M_F(X_{m,u}, A, \alpha, \varepsilon) \le \varlimsup_{\tau\to\infty} \sum_{T=\tau}^{+\infty} \theta^T = 0.$$

Therefore, $\alpha > P_F(A|_{X_{m,u}}, \varepsilon)$. Now let

$$r_F = \inf_{\mu\in\mathcal{M}} \lim_{t\to\infty} F\left(\frac{1}{t}\int_X A_t\, d\mu\right) \quad \text{and} \quad s_F = \sup_{\mu\in\mathcal{M}} \lim_{t\to\infty} F\left(\frac{1}{t}\int_X A_t\, d\mu\right).$$

For each $u \in [r_F, s_F]$ there exist $u_1, \ldots, u_r \in \mathbb{R}$ such that $|u - u_j| < \delta$ for some $j \in \{1, \ldots, r\}$. Then

$$X = \bigcup_{m\in\mathbb{N}} \bigcup_{j=1}^{r} X_{m,u_j}.$$

Applying the second property in Proposition 3, we obtain

$$\begin{aligned}
K + 3\delta + \lim_{\varepsilon\to 0} \bar{\gamma}(\varepsilon) &\ge \lim_{\varepsilon\to 0} \sup_{m,j} P_F(A|_{X_{m,u_j}}, \varepsilon) \\
&= \lim_{\varepsilon\to 0} P_F(A|_X, \varepsilon) = P_F(A|_X) = P_F(A).
\end{aligned}$$

Together with property (3), this gives that $P_F(A) \le K + 3\delta$ and it follows from the arbitrariness of δ that $P_F(A) \le K$.

Step 2. We proceed with the proof of the reverse inequality

$$P_F(A) \geq \sup_{\mu\in\mathcal{M}}\left\{h_\mu(\Phi) + \lim_{t\to\infty} F\left(\frac{1}{t}\int_X A_t\,d\mu\right)\right\}.$$

Lemma 3. *For each ergodic $\mu \in \mathcal{M}$, we have*

$$P_F(A) \geq h_\mu(\Phi) + \lim_{t\to\infty} F\left(\frac{1}{t}\int_X A_t\,d\mu\right).$$

Proof of the lemma. We proceed in a similar manner to that in the proof of Lemma 2.4 in [5]. Given $\varepsilon > 0$, there exist $\delta \in (0, \varepsilon)$, a measurable partition $\xi = \{C_1, \ldots, C_m\}$ of X and an open cover $\mathcal{V} = \{V_1, \ldots, V_k\}$ of X for some $k \geq m$ such that:

- $\operatorname{diam} C_j \leq \varepsilon$, $\overline{V_i} \subset C_i$ and $\mu(C_i \setminus V_i) < \delta^2$ for $i = 1, \ldots, m$;
- the set $E = \bigcup_{i=m+1}^k V_i$ satisfies $\mu(E) < \delta^2$.

Let ν be a measure in the ergodic decomposition of μ with respect to the time-1 map ϕ_1 (note that μ need not be ergodic with respect to ϕ_1). Given $x \in X$ and $n \in \mathbb{N}$, let $s_n(x)$ be the number of integers $l \in [0, n)$ such that $\phi_l(x) \in E$. By the uniform continuity of F, there exist a set D with $\nu(D) \geq 1 - 2\delta$ and $N \in \mathbb{N}$ such that

$$\frac{s_n(x)}{n} < 2\delta, \qquad \left|F\left(\frac{A_n(x)}{n}\right) - \lim_{t\to\infty} F\left(\frac{1}{t}\int_X A_t\,d\mu\right)\right| < \delta \tag{10}$$

and

$$\nu(\xi_n(x)) \leq \exp[(-h_\nu(\phi_1, \xi) + \delta)n] \tag{11}$$

for all $x \in D$ and $n > N$, where $\xi_n = \bigvee_{j=0}^n \phi_{-j}\xi$.

Now let Δ be a Lebesgue number of the cover $\mathcal{V}$ and take $\rho > 0$ with $2\rho < \Delta$. Given $\alpha \in \mathbb{R}$, by definition one can take $N_1 > N$ such that for each $n > N_1$ there exists $\Gamma \subset X \times [n, \infty)$ covering X such that

$$\left|\sum_{(x,t)\in\Gamma} \exp(A_F(x, t, \rho) - \alpha t) - M_F(X, A, \alpha, \rho)\right| < \delta. \tag{12}$$

Using (10) and (11), one can proceed as in the proof of Lemma 12 in [3] to obtain

$$\operatorname{card}\Gamma_l \geq \nu(B_l \cap D)\exp\bigl[h_\nu(\phi_1, \xi)l - (1 + 2\log\operatorname{card}\xi)l\delta\bigr] \tag{13}$$

for $l \in \mathbb{N}$, where $B_l = \bigcup_{(x,t)\in\Gamma_l} B_t(x, \rho)$ and

$$\Gamma_l = \{(x, t) \in \Gamma : B_l(x, \rho) \cap D \neq \emptyset\}.$$

It follows from the second inequality in (10) that

$$\sup_{y\in B_l(x,\rho)} lF\left(\frac{A_l(y)}{l}\right) \geq l\left(\lim_{t\to\infty} F\left(\frac{1}{t}\int_X A_t\,d\mu\right) - \delta\right) - \bar{\gamma}_l(A, \rho) \tag{14}$$

for every $l > N_1$ and $(x, t) \in \Gamma_l$.

Note that

$$\begin{aligned}&\sum_{(x,t)\in\Gamma} \exp(A_F(x, t, \rho) - \alpha t)\\ &\geq \sum_{l=N_1}^\infty \sum_{(x,t)\in\Gamma_l} \exp\left[\sup_{y\in B_l(x,\rho)} lF\left(\frac{A_l(y)}{l}\right) - \alpha l\right].\end{aligned}$$

Applying (13) and (14) we obtain

$$\sum_{(x,t)\in\Gamma} \exp(A_F(x,t,\rho)-\alpha t) \geq$$

$$\geq \sum_{l=N_1}^{\infty} \nu(B_l\cap D)\exp\left[l\left(b-\frac{\bar{\gamma}_l(A,\rho)}{l}-\alpha\right)-2\delta l(1+\log\operatorname{card}\xi)\right],$$

where

$$b=h_\nu(\phi_1,\xi)+\lim_{t\to\infty} F\left(\frac{1}{t}\int_X A_t\,d\mu\right).$$

Now assume that N_1 is so large that

$$\frac{\bar{\gamma}_l(A,\rho)}{l} \leq \varlimsup_{t\to\infty}\frac{\bar{\gamma}_t(A,\rho)}{t}+\delta \quad \text{for } l\geq N_1$$

and take

$$\alpha < b-\varlimsup_{t\to\infty}\frac{\bar{\gamma}_t(A,\rho)}{t}.$$

One can also take δ sufficiently small such that

$$\alpha < b-\varlimsup_{t\to\infty}\frac{\bar{\gamma}_t(A,\rho)}{t}-2\delta(1+\log\operatorname{card}\xi)-\delta.$$

Then

$$\sum_{(x,t)\in\Gamma} \exp(A_F(x,t,\rho)-\alpha t) \geq \sum_{l=N_1}^{\infty}\nu(B_l\cap D)\geq \nu(D)\geq 1-2\delta.$$

Together with (12), we obtain $M_F(X,A,\alpha,\rho)>1-3\delta>0$, which gives

$$P_F(A,\rho)\geq h_\nu(\phi_1,\xi)+\lim_{t\to\infty} F\left(\frac{1}{t}\int_X A_t\,d\mu\right)-\varlimsup_{t\to\infty}\frac{\bar{\gamma}_t(A,\rho)}{t}.$$

Since

$$\lim_{\rho\to 0}\varlimsup_{t\to\infty}\frac{\bar{\gamma}_t(A,\rho)}{t}=0,$$

one can proceed as in [5] to conclude that

$$P_F(A)\geq h_\nu(\phi_1)+\lim_{t\to\infty} F\left(\frac{1}{t}\int_X A_t\,d\mu\right).$$

Finally, integrating with respect to ν we obtain

$$\begin{aligned}P_F(A) &\geq h_\mu(\phi_1)+\lim_{t\to\infty} F\left(\frac{1}{t}\int_X A_t\,d\mu\right)\\ &= h_\mu(\Phi)+\lim_{t\to\infty} F\left(\frac{1}{t}\int_X A_t\,d\mu\right),\end{aligned}$$

which completes the proof of the lemma. □

Now let $\mu\in\mathcal{M}$ be an arbitrary measure. It follows from Lemma 3 and the assumption that (Φ,A) has an abundance of ergodic measures, that given $\varepsilon>0$ there exists $\nu\in\mathcal{M}_{\mathrm{erg}}$ such that

$$\begin{aligned}P_F(A) &\geq h_\nu(\Phi)+\lim_{t\to\infty} F\left(\frac{1}{t}\int_X A_t\,d\nu\right)\\ &\geq h_\mu(\Phi)+\lim_{t\to\infty} F\left(\frac{1}{t}\int_X A_t\,d\mu\right)-\varepsilon.\end{aligned}$$

Since μ and ε are arbitrary, we obtain

$$P_F(A) \geq \sup_{\mu \in \mathcal{M}} \left\{ h_\mu(\Phi) + \lim_{t\to\infty} F\left(\frac{1}{t}\int_X A_t\, d\mu\right)\right\}.$$

This completes the proof of the theorem. □

Under the hypotheses of Theorem 4 one can also obtain a variational principle over the ergodic invariant measures, that is, if the pair (Φ, A) has an abundance of ergodic measures then

$$P_F(A) = \sup_{\mu \in \mathcal{M}_{\mathrm{erg}}} \left\{ h_\mu(\Phi) + \lim_{t\to\infty} F\left(\frac{1}{t}\int_X A_t\, d\mu\right)\right\}.$$

4. Existence and characterization of equilibrium measures

In this section we discuss the existence and characterization of equilibrium measures. We shall use some machinery coming from the multifractal analysis developed in [4].

4.1. **Existence of equilibrium measures.** Following Theorem 4, we say that a measure $\mu \in \mathcal{M}$ is an *equilibrium measure for the pair* (F, A) with respect to the flow Φ if

$$P_F(A) = h_\mu(\Phi) + \lim_{t\to\infty} F\left(\frac{1}{t}\int_X A_t\, d\mu\right).$$

Theorem 5. *Let Φ be a continuous flow on a compact metric space X such that the entropy map $\mu \mapsto h_\mu(\Phi)$ is upper-semicontinuous. Moreover, take $A \in \mathcal{A}(X)^d$ and let $F\colon \mathbb{R}^d \to \mathbb{R}$ be a continuous function. If the pair (Φ, A) has an abundance of ergodic measures, then there exists at least one equilibrium measure for (F, A).*

Proof. Since the map $\mu \mapsto h_\mu(\Phi)$ is upper-semicontinuous, and the function F as well as the map $\mu \mapsto \lim_{t\to\infty} \frac{1}{t}\int_X A_t\, d\mu$ are continuous (see for example the proof of Theorem 2.5 in [5]), the map

$$\mathcal{M} \ni \mu \mapsto h_\mu(\Phi) + \lim_{t\to\infty} F\left(\frac{1}{t}\int_X A_t\, d\mu\right)$$

is upper-semicontinuous. Hence, the compactness of $\mathcal{M}$ guarantees the existence of a measure $\mu_A \in \mathcal{M}$ such that

$$\sup_{\mu \in \mathcal{M}} \left\{ h_\mu(\Phi) + \lim_{t\to\infty} F\left(\frac{1}{t}\int_X A_t\, d\mu\right)\right\} = h_{\mu_A}(\Phi) + \lim_{t\to\infty} F\left(\frac{1}{t}\int_X A_t\, d\mu_A\right).$$

Thus, it follows from Theorem 4 that μ_A is an equilibrium measure for the pair (F, A). □

4.2. **Characterization of equilibrium measures.** Given a pair (Φ, A), we consider the set

$$L(A) = \left\{ \lim_{t\to\infty} \frac{1}{t}\int_X A_t\, d\mu : \mu \in \mathcal{M}\right\} \subset \mathbb{R}^d.$$

Since the map $\mu \mapsto \lim_{t\to\infty} \frac{1}{t}\int_X A_t\, d\mu$ is continuous and $\mathcal{M}$ is compact and connected, the set $L(A)$ is a compact and connected subset of $\mathbb{R}^d$. For each $z \in \mathbb{R}^d$, we also consider the level sets

$$\mathcal{M}(z) = \left\{\mu \in \mathcal{M} : \lim_{t\to\infty} \frac{1}{t}\int_X A_t\, d\mu = z\right\}$$

and

$$C_z(A) = \left\{x \in X : \lim_{t\to\infty} \frac{A_t(x)}{t} = z\right\}.$$

We say that a pair (Φ, A) is C^1*-regular* if the following properties hold:

1. each family in $\operatorname{span}\{a^1, \dots, a^d, 1\}$ has a unique equilibrium measure for the nonadditive topological pressure and $\operatorname{int} L(A) \neq \emptyset$;
2. the map $\mu \mapsto h_\mu(\Phi)$ is upper-semicontinuous.

Notice that by Proposition 6 in [4], if the pair (Φ, A) is C^1-regular, then for each $i \in \{1, \dots, d\}$ the function $s \mapsto P(sa^i)$ is C^1 for each $s \in \mathbb{R}$ (actually, this is one of the reasons behind the name C^1-regular in the definition). Moreover, it follows from Theorem 1.2 in [5] that if $\Lambda \subset X$ is a locally maximal hyperbolic set for a topologically mixing C^1 flow Φ and each family a^i in A has bounded variation, then (Φ, A) is C^1-regular.

Given a continuous function $F\colon \mathbb{R}^d \to \mathbb{R}$, we consider the set

$$K(F, A) = \left\{\lim_{t\to\infty} \frac{1}{t}\int_X A_t\, d\mu : \mu \text{ is an equilibrium measure for } (F, A)\right\}.$$

We also define a function $h\colon L(A) \to \mathbb{R}$ by

$$h(z) = \sup\{h_\mu(\Phi) : \mu \in \mathcal{M}(z)\}. \tag{15}$$

Finally, let $\mathcal{E}(X) \subset \mathcal{A}(X)$ be the set of all families with a unique equilibrium measure. The following theorem is our main result.

Theorem 6. *Let Φ be a continuous flow on a compact metric space X and take $A \in \mathcal{A}(X)^d$ such that the pair (Φ, A) is C^1-regular. For each continuous function $F\colon \mathbb{R}^d \to \mathbb{R}$ the following properties hold:*

1. *$K(F, A)$ is a nonempty compact set;*
2. *$K(F, A)$ is the set of maximizers of the function $z \mapsto h(z) + F(z)$;*
3. *if $K(F, A) \subset \operatorname{int} L(A)$, then the equilibrium measures for (F, A) are the elements of the set $\{\nu_z : z \in K(F, A)\}$, where each ν_z satisfies the following:*
   - *ν_z is ergodic;*
   - *ν_z is the unique invariant measure in $\mathcal{M}(z)$ with $\nu_z(C_z(A)) = 1$ such that $h_{\nu_z}(\Phi) = h(z)$;*
   - *ν_z is the unique equilibrium measure for the almost additive family of continuous functions given by*

     $$(b_z)_t = \langle q(z), A_t - z\rangle - h(z)t$$

     for some $q(z) \in \mathbb{R}^d$.

Proof. We prove the theorem through some lemmas, which are obtained using the multifractal analysis result in Theorem 2.

Lemma 4. *$K(F, A)$ is a nonempty compact subset of $L(A)$.*

Proof of the lemma. Let $(z_n)_{n\in\mathbb{N}}$ be a sequence in $K(F,A)$ converging to a point $z \in L(A)$. For each $n \in \mathbb{N}$ there exists an equilibrium measure $\mu_n \in \mathcal{M}$ for (F,A) such that

$$z_n = \lim_{t\to\infty} \frac{1}{t} \int_X A_t \, d\mu_n.$$

Eventually passing to a subsequence, one can assume that there exists $\mu \in \mathcal{M}$ such that $\mu_n \to \mu$ in the weak* topology. Since the entropy map $\mu \mapsto h_\mu(\Phi)$ is upper-semicontinuous, and $\mu \mapsto \lim_{t\to\infty} \frac{1}{t}\int_X A_t \, d\mu$ and $F\colon \mathbb{R}^d \to \mathbb{R}$ are continuous, we obtain

$$\begin{aligned} P_F(A) &= \overline{\lim_{n\to\infty}} \left[h_{\mu_n}(\Phi) + \lim_{t\to\infty} F\left(\frac{1}{t} \int_X A_t \, d\mu_n \right) \right] \\ &\le h_\mu(T) + \lim_{t\to\infty} F\left(\frac{1}{t} \int_X A_t \, d\mu \right). \end{aligned}$$

This readily implies that $\mu \in \mathcal{M}$ is also an equilibrium measure for (F,A). Since

$$z = \lim_{t\to\infty} \frac{1}{t} \int_X A_t \, d\mu,$$

we conclude that $z \in K(F,A)$. Hence, $K(F,A)$ is a closed subset of $\mathbb{R}^d$. Since $K(F,A)$ is also bounded, we conclude that it is a compact set. Moreover, by Theorem 5 we have $K(F,A) \ne \emptyset$. $\square$

Lemma 5. *For each $z \in \operatorname{int} L(A)$ there exists an ergodic measure $\nu_z \in \mathcal{M}$ such that*

$$\lim_{t\to\infty} \frac{1}{t} \int_X A_t d\nu_z = z \quad \text{and} \quad \nu_z(C_z(A)) = 1.$$

Moreover, ν_z is the unique equilibrium measure for some almost additive family $b_z \in \mathcal{E}(X)$.

Proof of the lemma. Notice that by the definition of the function h in (15) and the first item of Theorem 2, we have

$$h(z) = h_{\text{top}}(\Phi | C_z(A)) \quad \text{for } z \in \operatorname{int} L(A).$$

For each $z \in \operatorname{int} L(A)$, consider the family $Q = (Q_t)_{t\ge 0}$ given by

$$Q_t = \langle q, A_t - tz \rangle - h(z)t$$

and define $\Xi_z(q) = P(Q)$. By Lemmas 10 and 11 in [4], we have that

$$\inf_{q\in\mathbb{R}^d} \Xi_z(q) \ge 0 \quad \text{for } z \in L(A),$$

$$\inf_{q\in\mathbb{R}^d} \Xi_z(q) = 0 \quad \text{for } z \in \operatorname{int} L(A),$$

and there exists at least one point $q(z) \in \mathbb{R}^d$ such that $\Xi_z(q(z)) = 0$. Since

$$\lim_{t\to\infty} \frac{1}{t} \int_X Q_t \, d\mu = \lim_{t\to\infty} \frac{1}{t} \int_X \langle q, A_t \rangle \, d\mu - (\langle q, z \rangle + h(z))$$

for every $\mu \in \mathcal{M}$ and $\operatorname{span}\{a^1, \ldots, a^d, 1\} \subset \mathcal{E}(X)$, we obtain that $Q \in \mathcal{E}(X)$ for each $q \in \mathbb{R}^d$. Hence, by Proposition 6 in [4] the map $q \mapsto \Xi_z(q)$ is of class C^1, and we conclude that $\partial_q \Xi_z(q(z)) = 0$.

Now let ν_z be the unique equilibrium measure for the family b_z given by

$$(b_z)_t = \langle q(z), A_t - tz \rangle - h(z)t.$$

Again by Proposition 6 in [4], the measure ν_z is ergodic and

$$\lim_{t\to\infty}\frac{1}{t}\int_X (A_t - tz)d\nu_z = \partial_q \Xi_z(q(z)) = 0,$$

which implies that

$$\lim_{t\to\infty}\frac{1}{t}\int_X A_t\, d\nu_z = z.$$

Moreover, proceeding as in the proof of Theorem 8 in [4], we can also verify that $\nu_z(C_z(A)) = 1$. $\square$

Lemma 6. *For each $z \in L(A)$ there exists $\mu \in \mathcal{M}$ such that $h(z) = h_\mu(\Phi)$. In addition, when $z \in \operatorname{int} L(A)$ this measure is unique and coincides with ν_z.*

Proof of the lemma. Take $z \in L(A)$. By definition, there exists a measure $\mu \in \mathcal{M}$ such that $z = \lim_{t\to\infty}\frac{1}{t}\int_X A_t\, d\mu$ and so $\mathcal{M}(z) \neq \emptyset$. Since the entropy map $\mu \mapsto h_\mu(\Phi)$ is upper-semicontinuous and the set $\mathcal{M}(z)$ is compact, there exists a measure $\mu \in \mathcal{M}(z)$ maximizing the metric entropy.

Now take $z \in \operatorname{int} L(A)$ and let $\mu \in \mathcal{M}(z)$ be a measure maximizing the metric entropy. By Lemma 5, there exists an ergodic measure $\nu_z \in \mathcal{M}(z)$ that is the unique equilibrium measure for the almost additive continuous function b_z. Since

$$\lim_{t\to\infty}\frac{1}{t}\int_X A_t\, d\mu = \lim_{t\to\infty}\frac{1}{t}\int_X A_t\, d\nu_z,$$

one can easily verify that

$$\lim_{t\to\infty}\frac{1}{t}\int_X (b_z)_t d\mu = \lim_{t\to\infty}\frac{1}{t}\int_X (b_z)_t\, d\nu_z.$$

Then

$$h_\mu(\Phi) + \lim_{t\to\infty}\frac{1}{t}\int_X (b_z)_t\, d\mu \geq h_{\nu_z}(\Phi) + \lim_{t\to\infty}\frac{1}{t}\int_X (b_z)_t d\nu_z = P(b_z),$$

which implies that μ is also an equilibrium measure for the family b_z. Since $b_z \in \mathcal{E}(X)$, we conclude that $\mu = \nu_z$. $\square$

Lemma 7. *$z \in K(F, A)$ if and only if z maximizes the function $E := h + F$.*

Proof of the lemma. First take $z \in L(A)$ maximizing E. By Lemma 6, there exists $\mu \in \mathcal{M}(z)$ such that $h(z) = h_\mu(\Phi)$. Then

$$\begin{aligned}
h_\mu(\Phi) + \lim_{t\to\infty} F\left(\frac{1}{t}\int_X A_t\, d\mu\right) &= h(z) + F(z)\\
\geq \sup_{w\in L(A)}\ \sup_{\eta\in\mathcal{M}(w)} &\left\{h_\eta(\Phi) + F\left(\lim_{t\to\infty}\frac{1}{t}\int_X A_t\, d\eta\right)\right\}\\
= \sup_{\eta\in\mathcal{M}} &\left\{h_\eta(\Phi) + \lim_{t\to\infty} F\left(\frac{1}{t}\int_X A_t\, d\eta\right)\right\},
\end{aligned}$$

and $\mu \in \mathcal{M}$ is an equilibrium measure for (F, A). That is, $z \in K(F, A)$.

Conversely, assume that $z \in K(F, A)$. By definition, there exists an equilibrium measure $\mu \in \mathcal{M}$ for (F, A) such that $z = \lim_{t\to\infty}\frac{1}{t}\int_X A_t\, d\mu$.

Then

$$\begin{aligned}E(z) &= h(z) + F(z)\\ &\geq h_\mu(\Phi) + F\left(\lim_{t\to\infty}\frac{1}{t}\int_X A_t\,d\mu\right)\\ &= \sup_{\mu\in\mathcal{M}}\left\{h_\mu(\Phi) + \lim_{t\to\infty} F\left(\frac{1}{t}\int_X A_t\,d\mu\right)\right\} = \sup_{w\in L(A)} E(w),\end{aligned}$$

which completes the proof of the lemma. □

Items (1) and (2) are Lemmas 4 and 7, respectively. Now we establish item (3). By definition, for each $z \in K(F, A)$ there exists an equilibrium measure $\mu \in \mathcal{M}$ for (F, A) such that $\lim_{t\to\infty}\frac{1}{t}\int_X A_t\,d\mu = z$. When $K(F, A) \subset \operatorname{int} L(a)$, it follows from Lemmas 5 and 6 that μ is the unique measure in $\mathcal{M}(z)$ and that $\mu = \nu_z$, where ν_z is an ergodic measure that is the unique equilibrium measure for the family $b_z \in \mathcal{E}(X)$. □

Proposition 7. *Let (Φ, A) be a C^1-regular pair. Then the function h is upper-semicontinuous, finite and concave on $L(A)$. Moreover, h is continuous on* $\operatorname{int} L(A)$.

Proof. We first show that h is upper-semicontinuous. Take $z \in L(A)$ and consider a sequence $(z_n)_{n\in\mathbb{N}} \subset L(A)$ such that $z_n \to z$ when $n \to \infty$. By Lemma 6 and if necessary passing to a subsequence, for each $n \in \mathbb{N}$ there exists $\mu_n \in \mathcal{M}(z_n)$ such that $h(z_n) = h_{\mu_n}(\Phi)$ and $\mu_n \to \mu$ in the weak* topology when $n \to \infty$, for some measure $\mu \in \mathcal{M}$. We also have

$$\lim_{t\to\infty}\frac{1}{t}\int_X A_t\,d\mu = \lim_{n\to\infty}\lim_{t\to\infty}\frac{1}{t}\int_X A_t\,d\mu_n = \lim_{n\to\infty} z_n = z,$$

that is, $\mu \in \mathcal{M}(z)$. Since $\mu \mapsto h_\mu(\Phi)$ is upper-semicontinuous, we obtain

$$\varlimsup_{n\to\infty} h(z_n) = \varlimsup_{n\to\infty} h_{\mu_n}(\Phi) \leq h_\mu(\Phi) \leq h(z)$$

and so h is upper-semicontinuous on $L(A)$. The compactness of $L(A)$ together with the upper-semicontinuity of h on $L(A)$ and the fact that $\mathcal{M}(z) \neq \emptyset$ for each $z \in L(A)$, guarantee that h is finite on $L(A)$.

In order to prove that h is concave, take $z_1, z_2 \in L(A)$. By Lemma 6, there exist $\mu_1 \in \mathcal{M}(z_1)$ and $\mu_2 \in \mathcal{M}(z_2)$ such that $h_{\mu_1}(\Phi) = h(z_1)$ and $h_{\mu_2}(\Phi) = h(z_2)$. Since the metric entropy is affine, for each $t \in [0, 1]$ we have

$$\begin{aligned}h(tz_1 + (1-t)z_2) &\geq h_{t\mu_1+(1-t)\mu_2}(\Phi)\\ &= th_{\mu_1}(\Phi) + (1-t)h_{\mu_2}(\Phi)\\ &= th(z_1) + (1-t)h(z_2),\end{aligned}$$

which implies that h is concave on $L(A)$, as desired. The continuity of h on $\operatorname{int} L(A)$ follows directly from item 3 in Theorem 2. □

Remark 1. In the one-dimensional case $d = 1$ with $A = a$, we actually obtain that h is continuous on $L(a)$. In fact, since h is upper-semicontinuous and finite on the interval $L(a)$, one can easily verify that h is also continuous on $\partial L(a)$. Therefore, h is continuous on the closed interval $L(a)$. In the multidimensional case, the function h may be discontinuous on $\partial L(A)$ (see the recent work [30]).

We establish one further result concerning the uniqueness of equilibrium measures.

Theorem 8. *Let (Φ, A) be a C^1-regular pair and let $F\colon \mathbb{R}^d \to \mathbb{R}$ be a continuous function that is strictly concave on $L(A)$. If $E = h + F$ attains its maximum on* $\operatorname{int} L(A)$*, then there exists a unique equilibrium measure for (F, A) and this equilibrium measure is ergodic.*

Proof. By Proposition 7, the function h is upper-semicontinuous on $L(A)$. This implies that E is also upper-semicontinuous on $L(A)$. By the compactness of $L(A)$, there exists at least one point in $L(A)$ maximizing E. By assumption, there is no maximizer in $\partial L(A)$. Since h is concave and F is assumed to be strictly concave, the function E is also strictly concave. Then there exists a unique point $z^* \in \operatorname{int} L(A)$ such that $E(z^*)$ is a maximum. It follows from Theorem 6 that

$$K(F, A) = \{z^*\} \subset \operatorname{int} L(A)$$

and that ν_{z^*} is the unique equilibrium measure for (F, A), where ν_{z^*} is an ergodic measure that is the unique equilibrium for the family b_z given by item 3 of Theorem 6. ☐

We note that the concavity of F alone does not guarantee the uniqueness of equilibrium measures for (F, A). In fact, (F, A) may have an uncountable number of equilibrium measures (see items (a) and (b) in Figure 1). When F is strictly concave, it may happen that the maximizer of E lies in $\partial L(A)$, in which case we are not able to apply Theorem 6 to characterize the equilibrium measures (see item (d) in Figure 1). The conditions in Theorem 8 are verified in item (c) in Figure 1, where thus we have a unique equilibrium measure for (F, A).

5. The additive case

In this section we consider the particular case when the family of functions is additive. Given a continuous function $a\colon X \to \mathbb{R}$, let

$$a_t(x) = \int_0^t (a \circ \phi_s)(x)ds \quad \text{for } x \in X \text{ and } t \geq 0.$$

Clearly, the family $a = (a_t)_{t\geq 0}$ satisfies

$$a_{t+s}(x) = a_t(x) + a_s(\phi_s(x)) \quad \text{for } x \in X \text{ and } t, s \geq 0.$$

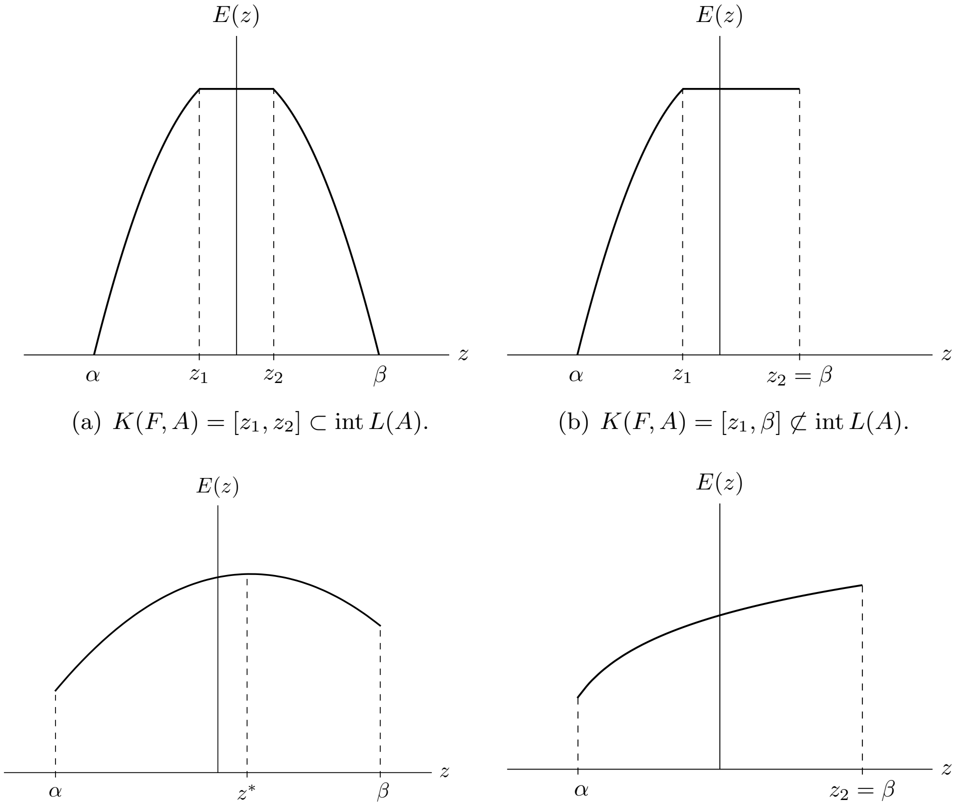
When compared to the almost additive setup, in this case we have more tools to deal with the problem of existence and uniqueness of nonlinear equilibrium measures.

We say that a function $a\colon X \to \mathbb{R}$ is *cohomologous* to another function $b\colon X \to \mathbb{R}$ (with respect to the flow Φ) if there exists a measurable bounded function $q\colon X \to \mathbb{R}$ such that

$$a(x) - b(x) = \lim_{s\to 0} \frac{q(\phi_s(x)) - q(x)}{s}$$

for every $x \in X$. Consider the families

$$a_t^i = \int_0^t (a^i \circ \phi_s)ds \quad \text{for } i \in \{1, \dots, d\}.$$



(a) $K(F,A)=[z_1,z_2]\subset \operatorname{int} L(A)$. (b) $K(F,A)=[z_1,\beta]\not\subset \operatorname{int} L(A)$.

(c) $K(F,A)=\{z^*\}\subset \operatorname{int} L(A)$. (d) $K(F,A)=\{\beta\}\not\subset \operatorname{int} L(A)$.

FIGURE 1. One-dimensional concave case: some possible shapes for the function $E\colon L(A):=[\alpha,\beta]\to\mathbb{R}$.

By Kingman's subadditive ergodic theorem we have

$$L(A)=\left\{\int_X A\,d\mu : \mu\in\mathcal{M}\right\} \quad\text{and}\quad \mathcal{M}(z)=\left\{\mu\in\mathcal{M} : \int_X A\,d\mu=z\right\},$$

where A is the vector of functions given by $A(x)=(a^1(x),\dots,a^d(x))$ for $x\in X$. Note that when a^i is cohomologous to a constant $c_i\in\mathbb{R}$ for each $i\in\{1,\dots,d\}$, we have $\int_X A\,d\mu=(c_1,\dots,c_d)$ for every $\mu\in\mathcal{M}$. This implies that $\operatorname{int} L(A)=\emptyset$ and $\mathcal{M}(z)=\emptyset$ for $z\neq(c_1,\dots,c_d)$.

In the additive case, inspired by [14], we say that a pair (Φ,A) is C^r-*regular* if for each $i\in\{1,\dots,d\}$ the function a^i is not cohomologous to a constant and the following properties hold:

1. each function in $\operatorname{span}\{a^1,\dots,a^d,1\}$ has a unique equilibrium measure for the classical topological pressure;
2. for each $z\in\operatorname{int} L(A)$ the map $\mathbb{R}^d\ni q\mapsto P(\langle q,A-z\rangle)$ is finite, C^r, strictly convex, and for $r\geq 2$ the second derivative matrix
$$\partial_q^2 P(\langle q,A-z\rangle)$$
is positive definite for every $q\in\mathbb{R}^d$;
3. the map $\mu\mapsto h_\mu(\Phi)$ is upper-semicontinuous.

In contrast to what happens in the nonadditive case, in the notion of regular pair for the additive setup we can talk about C^r-regular pairs with $r > 1$, where $r = \omega$ is the real analytic case. This happens because of the following result, which follows from Proposition 9 and Lemma 1 in [2].

Proposition 9. *Let $\Lambda \subset X$ be a locally maximal hyperbolic set for a C^1 flow Φ such that $\Phi|_\Lambda$ is topologically mixing. Then the following properties hold:*

1. *the entropy map $\mu \mapsto h_\mu(\Phi)$ is upper-semicontinuous;*
2. *each Hölder continuous function $a\colon X \to \mathbb{R}$ has a unique equilibrium measure;*
3. *given Hölder continuous functions $a, b\colon X \to \mathbb{R}$, the pressure function $\mathbb{R} \ni t \mapsto P(ta + b)$ is real analytic; moreover, for each $t \in \mathbb{R}$ we have*
$$\frac{d^2}{dt^2} P(ta + b) \geq 0,$$
with equality if and only if a is cohomologous to a constant;
4. *if every function in $A = (a^1, \ldots, a^d)$ is Hölder continuous, then the second derivative matrix*
$$\partial_q^2 P(\langle q, A - z\rangle - p)$$
is positive definite for every $z \in \operatorname{int} L(A)$, $q \in \mathbb{R}^d$ and $p \in \mathbb{R}$; in particular, the map $\mathbb{R}^d \ni q \mapsto P(\langle q, A - z\rangle)$ is strictly convex for every $z \in \operatorname{int} L(A)$.

Notice that by Proposition 9 the pair $(\Phi|_\Lambda, A)$ is C^ω-regular when $\Lambda \subset X$ is a locally maximal hyperbolic set for a topologically mixing C^1 flow and each $a^i\colon \Lambda \to \mathbb{R}$ is a Hölder continuous function not cohomologous to a constant.

Proposition 10. *If the pair (Φ, A) is C^r-regular, then the map $h|_{\operatorname{int} L(A)}$ is C^{r-1}. If (Φ, A) is C^ω-regular, then $h|_{\operatorname{int} L(A)}$ is real analytic. Moreover, $h\colon \operatorname{int} L(A) \to \mathbb{R}$ is strictly concave.*

Proof. By Theorem 10 in [2], if the pair (Φ, A) is C^r-regular, then the entropy spectrum
$$\operatorname{int} L(A) \ni z \mapsto h_{\text{top}}(\Phi|_{C_z(A)}) \tag{16}$$
is C^{r-1}. Moreover, if (Φ, A) is C^ω, then the entropy spectrum in (16) is also C^ω. Since the first item of Theorem 2 implies that $h(z) = h_{\text{top}}(\Phi|_{C_z(A)})$ for every $z \in \operatorname{int} L(A)$, we obtain the desired regularity of h.

Now assume $0 \in \operatorname{int} L(A)$, without loss of generality. By hypothesis, the map $\mathbb{R}^d \ni q \mapsto P(\langle q, A\rangle)$ is strictly convex, C^r and finite. Moreover, we have
$$P(\langle q, A\rangle) = \sup_{z \in L(A)} [h(z) + \langle q, z\rangle].$$
Hence, since $\operatorname{int} L(A) \neq \emptyset$, it follows directly from Theorem 11.13 in [25] that h is strictly concave on $\operatorname{int} L(A)$. ☐

Note that in order to prove Theorem 6 we do not need that the pair (Φ, A) is C^r-regular as defined in this section. In fact the second item in the definition is only used to establish the regularity and strict concavity of the function h.

Based on the strict concavity of h, we are able to improve Theorem 8 in the additive setup, by requiring F to be only concave instead of strictly concave.

Theorem 11. *Let (Φ, A) be a C^r-regular pair and let $F\colon \mathbb{R} \to \mathbb{R}$ be a continuous function that is concave on $L(A)$. Then there exists a unique equilibrium measure for (F, A) and this equilibrium measure is ergodic.*

Example 1 (Uniqueness for the classical case)**.** When $r \geq 2$, $A = a$ and F is the identity map, which is a concave continuous function, we recover the uniqueness of equilibrium measures for the classical case (see [18]). In fact, let $F(z) = z$ and assume that the critical point z^* of E is contained in $\operatorname{int} L(a) = (\alpha, \beta)$. This implies that E attains its maximum on $\operatorname{int} L(a)$ and it follows from Theorem 6 that the ergodic measure ν_{z^*} is the unique equilibrium measure for (F, a).

Example 2 (The analytic case)**.** Let (Φ, a) be a C^r-regular pair with $r = \omega$ and assume that $F\colon \operatorname{int} L(a) \to \mathbb{R}$ is real analytic. By hypothesis, the function $h\colon \operatorname{int} L(a) \to \mathbb{R}$ is analytic and so $E\colon \operatorname{int} L(a) \to \mathbb{R}$ is also analytic. Proceeding as in the proof of Corollary 4.19 in [14], one can show that the function E has finitely many critical points, all of them in $\operatorname{int} L(a)$. It follows from Theorem 6 that (F, a) has finitely many equilibrium measures.

Notice that in Theorem 11 we did not require E to attain its maximum on $\operatorname{int} L(A)$. This is because we have the following property in the additive setup.

Proposition 12. *If the pair (Φ, A) is C^r-regular with $r \geq 2$, then the function $E = h + F$ does not attain its maximum on $\partial L(A)$.*

Proof. Since, by hypothesis, the map $\mathbb{R}^d \ni q \mapsto P(\langle q, A\rangle)$ is strictly convex (taking $0 \in \operatorname{int} L(A)$, without loss of generality) and

$$h(z) = \inf_{q \in \mathbb{R}^d} [P(\langle q, A\rangle) - \langle q, z\rangle] \quad \text{for } z \in L(A),$$

it follows from Theorem 26.3 in [24] that, in particular, $\|\nabla h(z)\| \to \infty$ when $z \to \partial L(A)$. Since h is concave, by the claim in the proof of Theorem 4.15 in [14], we guarantee that E cannot attain its maximum on $\partial L(A)$. $\square$

We note that in the almost additive setup we can only guarantee that the function h is continuous on $\operatorname{int} L(A)$, and we do not know if the function $E = h + F$ attains its maximum on the boundary of $L(A)$. We give a natural example that addresses this issue.

Example 3. Let Λ be a locally maximal hyperbolic set for a C^1 flow $\Phi = (\phi_t)_{t\in\mathbb{R}}$. Moreover, let $E^s(x)$ and $E^u(x)$ be the stable and unstable spaces at x. We say that Φ has *bounded distortion* if there exist constants $K_1 > 0$, $K_2 > 0$ and Hölder continuous functions $b^s, b^u\colon \Lambda \to \mathbb{R}$ such that

$$K_1\|v\| \exp \int_0^t (b^s \circ \phi_\tau)(x)\, d\tau \leq \|d_x\phi_t v\| \leq K_2\|v\| \exp \int_0^t (b^s \circ \phi_\tau)(x)\, d\tau$$

for $v \in E^s(x)$, and

$$K_1\|v\| \exp \int_0^t (b^u \circ \phi_\tau)(x)\, d\tau \le \|d_x\phi_t v\| \le K_2\|v\| \exp \int_0^t (b^u \circ \phi_\tau)(x)\, d\tau$$

for $v \in E^u(x)$. In this case one can easily verify that the families $a^s = (a^s_t)_{t\ge 0}$ and $a^u = (a^u_t)_{t\ge 0}$ given by

$$a^s_t(x) = \log \|d_x\phi_t|_{E^s(x)}\| \quad \text{and} \quad a^u_t(x) = \log \|d_x\phi_t|_{E^u(x)}\|$$

are almost additive with respect to Φ and satisfy

$$\lim_{t\to\infty} \frac{1}{t}\left\| a^s_t - \int_0^t (b^s \circ \phi_\tau)\, d\tau \right\|_\infty = 0$$

and

$$\lim_{t\to\infty} \frac{1}{t}\left\| a^u_t - \int_0^t (b^u \circ \phi_\tau)\, d\tau \right\|_\infty = 0.$$

Considering the families $A = (a^s, a^u)$ and the functions $B = (b^s, b^u)$, we have $L(A) = L(B)$ and $C_z(A) = C_z(B)$. This implies that the entropy functions $h_A\colon L(A) \to \mathbb{R}$ and $h_B\colon L(B) \to \mathbb{R}$ also coincide. Therefore, since E cannot attain its maximal value on $\partial L(B)$, because the pair (Φ, B) is C^ω-regular, it follows that E does not attain a maximum on $\partial L(A)$.

Inspired by the former example, as well as by work in [17] and some open problems formulated in [9], one can ask the following natural questions:

1. Given an almost additive family of continuous functions $a = (a_t)_{t\ge 0}$ with respect to a continuous flow $\Phi = (\phi_t)_{t\in\mathbb{R}}$ on a compact metric space X, is there any continuous function $b\colon X \to \mathbb{R}$ such that
$$\lim_{t\to\infty} \frac{1}{t}\left\| a_t - \int_0^t (b \circ \phi_s) ds \right\|_\infty = 0 \ ? \tag{17}$$
2. When $\Phi|_\Lambda$ is a hyperbolic flow and the family a has bounded variation, is there any Hölder continuous function $b\colon \Lambda \to \mathbb{R}$ satisfying (17)?

Besides the problem of characterization of nonlinear equilibrium measures as illustrated in Example 3, positive answers to these questions would be very interesting for some extensions of thermodynamic formalism, multifractal analysis and ergodic optimization for flows (see for example [9]).

6. Suspension flows

In this final section we consider briefly the particular case of suspension flows. Let X be a compact metric space, $T\colon X \to X$ a homeomorphism and $\tau\colon X \to \mathbb{R}^+$ a Lipschitz function. Consider the space

$$W = \big\{(x, s) \in X \times \mathbb{R} : 0 \le s \le \tau(x)\big\}$$

and let Y be the set obtained from W identifying $(x, \tau(x))$ with $(T(x), 0)$ for each $x \in X$. Then a certain distance introduced by Bowen and Walters in [13] makes Y a compact metric space. The *suspension flow* over T with height function τ is the flow $\Psi = (\psi)_{t\in\mathbb{R}}$ on Y with the maps $\psi_t\colon Y \to Y$ defined by $\psi_t(x, s) = (x, s + t)$.

Let μ be a T-invariant probability measure on X. One can show that μ induces a Ψ-invariant probability measure ν on Y such that

$$\int_\Lambda g\,d\nu = \frac{\int_X I_g\,d\mu}{\int_X \tau\,d\mu} \tag{18}$$

for any continuous function $g\colon Y \to \mathbb{R}$, where $I_g(x) = \int_0^{\tau(x)} (g \circ \psi_s)(x)\,ds$. Conversely, any Ψ-invariant probability measure ν on Y is of this form for some T-invariant probability measure μ on X. Abramov's entropy formula says that

$$h_\nu(\Psi) = \frac{h_\mu(T)}{\int_X \tau\,d\mu}. \tag{19}$$

Now consider continuous functions $a\colon Y \to \mathbb{R}$ and $F\colon \mathbb{R} \to \mathbb{R}$ such that F is nonlinear. It follows from (18) and (19) that

$$\begin{aligned} h_\mu(\Psi) + F\left(\int_Y a\,d\mu\right) &= \frac{h_\nu(T)}{\int_X \tau\,d\nu} + F\left(\frac{\int_X I_a\,d\nu}{\int_X \tau\,d\nu}\right) \\ &= \frac{h_\nu(T) + \left(\int_X \tau\,d\nu\right)F\left(\frac{\int_X I_a\,d\nu}{\int_X \tau\,d\nu}\right)}{\int_X \tau\,d\nu} \\ &= \frac{h_\nu(T) + G\left(\int_X \tau\,d\nu, \int_X I_a\,d\nu\right)}{\int_X \tau\,d\nu} \end{aligned}$$

for any probability measures ν and μ as above, where $G\colon \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is the function given by

$$G(z_1, z_2) = z_1 F\left(\frac{z_2}{z_1}\right).$$

Since $\tau > 0$, it follows that $P_G(\tau, I_a) = 0$ if and only if $P_F(a) = 0$, where $P_G(\tau, I_a)$ is the *higher-dimensional nonlinear topological pressure for* $(G, (\tau, I_a))$ with respect to T introduced in [6]. Hence, when $P_F(a) = 0$ the invariant measure μ is an equilibrium measure for (F, a) if and only if ν is an equilibrium measure for $(G, (\tau, I_a))$. Assuming that the function $a\colon Y \to \mathbb{R}$ is Hölder continuous, it follows from Proposition 18 in [8] that $I_a\colon X \to \mathbb{R}$ is also Hölder continuous. Moreover, if $P_F(a) = 0$ and the map $T\colon X \to X$ is either a topologically mixing subshift of finite time, a $C^{1+\varepsilon}$ expanding diffeomorphism or a $C^{1+\varepsilon}$ diffeomorphism with a locally maximal hyperbolic set, one can apply former results for dynamical systems with discrete time (see Theorem 7 in [6]).

The next example shows that given a continuous function F, we cannot always guarantee that there exists a function $a\colon X \to \mathbb{R}$ such that $P_F(a) = 0$.

Example 4. Let Φ be a continuous flow on a compact metric space Y and let $F\colon \mathbb{R} \to \mathbb{R}$ be a nonnegative continuous function. We assume that Φ has positive topological entropy, entropy density of ergodic measures, and that the map $\mathcal{M} \ni \mu \mapsto h_\mu(\Phi)$ is upper-semicontinuous. Then there exists a measure $\eta \in \mathcal{M}$ with $h_\eta(\Phi) = h_{\text{top}}(\Phi) > 0$ such that

$$P_F(a) = \sup_{\mu \in \mathcal{M}(\Phi)} \left\{ h_\mu(\Phi) + F\left(\int_Y a\,d\mu\right) \right\} \ge h_\eta(\Phi) + F\left(\int_Y a\,d\eta\right) > 0$$

for any continuous $a\colon Y \to \mathbb{R}$.

This example also shows, in general and in strong contrast to what happens in the classical setup, that even for suspension flows one cannot always reduce the nonlinear thermodynamic formalism for continuous time to the one for discrete time.

Finally, we give an example based on the Curie–Weiss model that provides a setup in which we can use the discrete time setup to study the continuous time problem.

Example 5. For $X = \{-1,1\}^{\mathbb{N}}$, let $T\colon X \to X$ be the shift map and define a function $\varphi\colon X \to \mathbb{R}$ by $\varphi(\omega) = \omega_0$, where $\omega = (\omega_0, \omega_1, \ldots) \in X$. We also consider the function $F\colon \mathbb{R} \to \mathbb{R}$ given by

$$F(z) = \frac{\beta}{2} z^2, \quad \text{where } \beta \geq 0 \text{ is a physical parameter.}$$

Now let $\tau = 1$ and consider the suspension flow Ψ over T with height function τ. It follows from [7] that there exists a function $a\colon Y \to \mathbb{R}$ such that $I_a|_X = \varphi$. For instance, in this case one can define the function a by

$$a(\psi_t(\omega)) = 12\varphi(\omega)[t^3 - 2t^2 + t] \quad \text{for } x \in X \text{ and } t \in [0,1].$$

Then

$$\begin{aligned} h_\mu(\Psi) + F\left(\int_Y a\,d\mu\right) &= h_\nu(T) + F\left(\int_X \varphi\,d\nu\right) \\ &= h_\nu(T) + \frac{\beta}{2}\left(\int_X \varphi\,d\nu\right)^2 \end{aligned}$$

for every T-invariant probability measure ν on X, where μ is the induced measure on Y. Therefore, $P_F(a)$ with respect to Ψ is equal to $P_F(\varphi)$ with respect to T, and ν is an equilibrium measure for (F, φ) if and only if μ is an equilibrium measure for (F, a). Hence, it follows from [15] that:

- for $0 \leq \beta \leq 1$ there exists a unique equilibrium measure for (F, a);
- for $\beta > 1$ there exist two equilibrium measures for (F, a).

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