NONLINEAR DIMENSION SPECTRA, FULL MEASURES AND NONTYPICAL POINTS

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ABSTRACT. We obtain nonlinear conditional variational principles for families of continuous potentials, with and without uniqueness of equilibrium states. This includes discussing the connection with the topological pressure, giving a detailed analysis of nonlinear level sets, and providing information on full measures. We apply our results in particular to describe the relation between the dimension spectrum and the classical topological pressure, to study the dimension of sets of nontypical points with respect to nonlinear perturbations of Birkhoff's averages and some multiple ergodic averages, and to provide a better understanding of intermediate dimension properties for maps with some hyperbolic behavior.

CONTENTS

1. INTRODUCTION

Our work is a further contribution to the (mathematical) thermodynamic formalism and multifractal analysis, with emphasis on the study of nonlinear conditional variational principles and their applications. The notion of topological pressure, which is the starting point for the thermodynamic formalism, was introduced in the 1970's by Ruelle in [33] for expansive maps and then by Walters in [37] in full generality. Since then the theory was developed profusely, and has many applications. The developments include a variational principle for the Kolmogorov–Sinai entropy, and the study of the existence and uniqueness of equilibrium and Gibbs measures, with emphasis on hyperbolic dynamics, among many other properties. There are applications of the thermodynamic formalism for example to dimension theory of dynamical systems and a rigorous development of multifractal analysis, again with emphasis on hyperbolic dynamics, and to statistical physics. There are

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also several extensions of the thermodynamic formalism. These include the nonadditive thermodynamic formalism, which was used to obtain dimension estimates for some classes of invariant sets, the subadditive thermodynamic formalism and its generalized variational principle with applications to entropy spectra, and the almost additive thermodynamic formalism with consequences for the construction of weak Gibbs measures on nonconformal repellers. We refer the reader to the books [1, 10, 24, 25, 29, 30, 34, 39] for the description of these and many further developments.

In this work, we are interested in extending and unifying some topics of multifractal analysis by using what is usually called the thermodynamic approach. The main aim of our work is bringing together the thermodynamic formalism and the nonlinear flavor of multifractal analysis. The later was initiated long ago, although rather intermittently. This includes for example the study of nonlinear relations between frequencies of digits and the dimension of sets defined in terms of these relations (see for example [6]). Often, the lack of alternative formulas for conditional variational principles cause considerable difficulties in making more explicit computations of dimensions. It is also one of our objectives to obtain such alternative formulas in the context of studying nonlinear relations. Interestingly (and perhaps rather surprisingly as well), our main results do not require and can in fact be considered independent of the new nonlinear thermodynamic formalism introduced in [12] (see also [4]).

Before proceeding to a more detailed presentation of our results, we detail briefly the main contributions of the present work:

1. We apply the thermodynamic approach to obtain nonlinear versions of conditional variational principles for $\mathbb{R}^d$-valued continuous potentials, with and without uniqueness of equilibrium states.
2. This description includes the connection with topological pressure, a detailed analysis of generalized level sets with divergent points and nonlinear level sets, and information on ergodic (nonlinear) full measures.
3. We apply our results to describe a relation between the dimension spectrum and the classical topological pressure, while also giving examples related to the frequency of symbols and digits in some number representations.
4. Moreover, we use the existence of ergodic nonlinear full measures to study the dimension of the sets of nontypical points with respect to nonlinear perturbations of Birkhoff's averages and some multiple ergodic averages.
5. Finally, we apply our results on the existence of ergodic nonlinear full measures to obtain a better understanding of more general intermediate dimension properties for maps with some hyperbolic behavior.

Now we give a detailed description of our main result. For basic notions and results, we refer the reader to Section 2 (we avoid detailing them here, which would require a rather extended introduction). These include in particular notions related to linear and nonlinear level sets, dimension theory, and the thermodynamic formalism.

Our main result—a nonlinear conditional variational principle—describes relations between nonlinear entropy and dimension spectra and the classical topological pressure. It is obtained through a correspondence between level sets with different limit points and nonlinear level sets, combined with results in [13]. Here we formulate the result only in the particular case of entropy spectra (see Theorem 3 for the general case of dimension spectra). Consider a map $T\colon X \to X$ of a compact metric space. Given a collection of real-valued continuous functions $\Phi = \{\phi_1, \dots, \phi_d\}$

on X, we consider the compact set

$$L = \left\{ \int_X \Phi \, d\mu : \mu \in \mathcal{M}(T) \right\} \quad \text{with} \quad \int_X \Phi \, d\mu = \left(\int_X \phi_1 \, d\mu, \ldots, \int_X \phi_d \, d\mu \right),$$

where $\mathcal{M}(T)$ denotes the set of T-invariant probability measures on X.

Theorem 1. *Assume that T has upper semicontinuous metric entropy and finite topological entropy, and that there exists a dense subspace $D(X)$ in the continuous functions such that every $\xi \in D(X)$ has a unique equilibrium measure. Given a continuous function $F \colon U \to \mathbb{R}^p$, where U is an open set containing L, for each $\beta \in \mathbb{R}^p$ with $F^{-1}\beta \cap L \subset \operatorname{int} L$ we have:*

1.
$$\begin{aligned} h(T|C_\beta^F) &= \sup_{\mu \in \mathcal{M}(T)} \left\{ h_\mu(T) : F\left(\int_X \Phi \, d\mu \right) = \beta \right\} \\ &= \sup_{\alpha \in F^{-1}\beta} \sup_{\mu \in \mathcal{M}(T)} \left\{ h_\mu(T) : \int_X \Phi \, d\mu = \alpha \right\} \\ &= \sup_{\alpha \in F^{-1}\beta} \sup_{\mu \in \mathcal{M}(T)} \{ h_\mu(T) : \mu \in \mathcal{M}_{C_\alpha}(T) \} \\ &= \sup_{\alpha \in F^{-1}\beta} \inf_{q \in \mathbb{R}^d} P(\langle q, \Phi - \alpha \rangle) = \sup_{\alpha \in F^{-1}\beta} h(|C_\alpha); \end{aligned}$$

2. *given $\varepsilon > 0$, there exists an ergodic measure $\nu \in \mathcal{M}(T)$ with $F(\int_X \Phi \, d\nu) = \beta$ and $\nu(C_\beta^F) = 1$ such that*
$$\left| h_\nu(T) - h(T|C_\beta^F) \right| < \varepsilon.$$

As far as we know, this result gives the first relation between the topological pressure and a general nonlinear conditional variational principle. The approximation via ergodic measures is crucial for studying nontypical points in the nonlinear case (see Section 7), for establishing some connections to multiple ergodic averages (see Section 9), and for extending intermediate entropy and dimension properties with respect to some hyperbolic systems (see Section 10). Following the developments in [5] and [7], we also obtain a sharper nonlinear conditional variational principle with all suprema replaced by maxima under more restrictive assumptions (see Theorem 6). We consider briefly the case of mixed spectra for conformal repellers in Section 3.3.

As further developments, in Section 4 we discuss the number and characterization of full measures. It turns out that when Φ has more than one potential, one may have parameters β with uncountably many nonlinear full measures. Furthermore, we consider the problem of the regularity of the nonlinear dimension spectrum in Section 5, and we explore some connections between the multifractal spectra of level sets with divergent points and the nonlinear thermodynamic formalism introduced in [12] to study the finer structure of the level sets in Section 6.

Concerning applications, in Section 7 we use the existence of nonlinear full measures to study the dimension of nonlinear irregular sets. In particular, this extends several results in [8]. The main element of our approach is the approximation of nonlinear full measures by distinguishing measures. Using Markov partitions, one could also obtain corresponding results for uniformly expanding and hyperbolic maps. We emphasize that some related nonlinear multifractal problems were discussed in [22], although using different methods. However, their approach cannot decide if some types of irregular sets have full topological entropy (see Remark 8 for details). Furthermore, in Section 8 we describe some applications of the nonlinear conditional variational principle to relations between frequencies of digits and the entropy and dimension spectra that they define.

In Section 9, combining our main theorem with results in [19] for the full shift, we obtain a relation between the Hausdorff dimension of level sets generated by some multiple ergodic averages, the nonlinear Hausdorff dimension spectrum and the classical topological pressure. Together with the existence of distinguishing and ergodic full measures, this relation also gives conditions for which the set of nontypical points with respect to multiple ergodic averages has full Hausdorff dimension.

Finally, inspired by recent work in [17], in Section 10 we study some relations between intermediate entropy and dimension properties of ergodic measures and their nonlinear counterparts. This is a further application of the existence of ergodic nonlinear full measures obtained in our work.

2. Basic notions and preliminaries

2.1. **Linear and nonlinear level sets.** Let X be a compact metric space and let $T\colon X \to X$ be a continuous map. We denote by $\mathcal{M}(T)$ the set of T-invariant probability measures on X. Consider a collection $\Phi = \{\phi_1, \ldots, \phi_d\}$ of continuous functions $\phi_i\colon X \to \mathbb{R}$ for $i = 1, \ldots, d$ and a continuous function $F\colon U \to \mathbb{R}^p$, where $U \subset \mathbb{R}^d$ is an open set containing the compact set

$$L = \left\{ \int_X \Phi\, d\mu : \mu \in \mathcal{M}(T) \right\} \quad \text{with} \quad \int_X \Phi\, d\mu = \left(\int_X \phi_1\, d\mu, \ldots, \int_X \phi_d\, d\mu \right).$$

Given a continuous function $\varphi\colon X \to \mathbb{R}$, we write $S_n\varphi = \sum_{k=0}^{n-1} \varphi \circ T^k$ for $n \in \mathbb{N}$. For each $\alpha \in \mathbb{R}^d$, we consider the *linear level set*

$$C_\alpha = \left\{ x \in X : \lim_{n\to\infty} \frac{S_n\Phi(x)}{n} = \alpha \right\}, \quad \text{where } S_n\Phi = (S_n\phi_1, \ldots, S_n\phi_d).$$

It is well known that $C_\alpha = \emptyset$ whenever $\alpha \notin L$. Furthermore, for each $\beta \in \mathbb{R}^p$ we consider the *nonlinear level set*

$$C_\beta^F = \left\{ x \in X : \lim_{n\to\infty} F\left(\frac{S_n\Phi(x)}{n} \right) = \beta \right\}.$$

Clearly, $C_\beta^F = \emptyset$ whenever $\beta \notin F(L) \subset \mathbb{R}^p$.

For each $j \in \{1, \ldots, d\}$, we denote by $S_\infty^j(x) \subset \mathbb{R}$ the set of limit points of the sequence $(S_n\phi_j(x)/n)_{n\in\mathbb{N}}$. Since these sequences are bounded and

$$\lim_{n\to\infty} \left[\frac{1}{n+1} S_{n+1}\phi_j(x) - \frac{1}{n} S_n\phi_j(x) \right] = 0$$

for all $x \in X$ and $j \in \{1, ..., d\}$, exactly one of the following alternatives holds:

1. $(S_n\phi_j(x)/n)_{n\in\mathbb{N}}$ converges and $S_\infty^j(x)$ is a singleton or
2. $(S_n\phi_j(x)/n)_{n\in\mathbb{N}}$ diverges and $S_\infty^j(x)$ is the closed interval

$$I_j(x) = \left[\liminf_{n\to\infty} \frac{1}{n} S_n\phi_j(x), \limsup_{n\to\infty} \frac{1}{n} S_n\phi_j(x) \right].$$

Moreover, let $S_\infty(x) \subset \mathbb{R}^d$ be the set of limit points of the sequence $(S_n\Phi(x)/n)_{n\in\mathbb{N}}$. For each $A \subset \mathbb{R}^d$, we also consider the more general (level) sets

$$C_A = \{ x \in X : S_\infty(x) \subset A \}. \tag{1}$$

Note that for $A = \{\alpha\}$ we have $C_{\{\alpha\}} = C_\alpha$. Clearly, $\bigcup_{\alpha\in A} C_\alpha \subseteq C_A$, but this inclusion may be proper since C_A may contain points $x \in X$ for which the sequence $(S_n\Phi(x)/n)_{n\in\mathbb{N}}$ diverges.

One can also consider the sets $\widetilde{S}_\infty(x) = S_\infty^1(x) \times \cdots \times S_\infty^d(x)$. Clearly $S_\infty(x) \subseteq \widetilde{S}_\infty(x)$ and in general this inclusion may be proper. For instance, for $\Phi = \{\phi_1, \phi_2\}$ with ϕ_1 cohomologous to ϕ_2, the set $S_\infty(x)$ is either a point or a diagonal line. On the other hand, for each $x \in X$ the set $\widetilde{S}_\infty(x)$ is the square $I_1(x) \times I_2(x)$. Finally, given $A \subset \mathbb{R}^d$, we consider the level sets

$$\widetilde{C}_A = \{x \in X : \widetilde{S}_\infty(x) \subset A\}.$$

For each $x \in X$ the set $\widetilde{S}_\infty(x) \subseteq I_1(x) \times \cdots \times I_d(x)$ is either a point, a hyperplane or a hyperrectangle. For instance, in dimension 1 each set $\widetilde{S}_\infty(x)$ is a singleton or a closed interval, while in dimension 2 it is a point, a vertical or horizontal line, or a rectangle (see Figure 1).

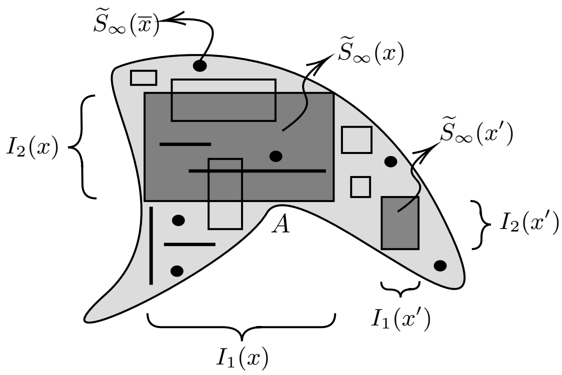

FIGURE 1. Possible types of sets $\widetilde{S}_\infty(x) \subset A \subset \mathbb{R}^2$.

The following example gives a general scenario where $C_A = \widetilde{C}_A$.

Example 1. Let $A \subset \mathbb{R}^d$ be a countable set. Then C_A and $\widetilde{C}_A$ are composed of points $x \in X$ for which the sequence $(S_n\Phi(x)/n)_{n\in\mathbb{N}}$ converges to some $\alpha \in A$, and one can see that $C_A = \bigcup_{\alpha\in A} C_\alpha = \widetilde{C}_A$. □

We give another example where the shape of the set $\widetilde{S}_\infty(x)$ plays a crucial role.

Example 2. Let $F\colon \mathbb{R}^2 \to \mathbb{R}$ be the quadratic function $F(x, y) = x^2+y^2$. For each $\beta \in \operatorname{int} L$, the set $A := F^{-1}\beta$ is the circle $S^1 \subset \mathbb{R}^2$ of radius $\sqrt{\beta}$ centered at $(0, 0)$. Since there are no lines nor rectangles contained in S^1, the set $\widetilde{C}_A$ is composed of points $x \in X$ for which $(S_n\Phi(x)/n)_{n\in\mathbb{N}}$ converges to some $\alpha \in A$. In this case A is an uncountable set, but we still have $\widetilde{C}_A = \bigcup_{\alpha\in A} C_\alpha$. □

We note that in general $\widetilde{C}_A \subset C_A$, and we may have $C_A \neq \bigcup_{\alpha\in A} C_\alpha$. When F is a nonlinear function and $F^{-1}\beta$ contains at least one hyperplane, it is not hard to find a dynamical system and continuous potentials such that $C_{F^{-1}\beta} \neq \bigcup_{\alpha\in F^{-1}\beta} C_\alpha$.

In this work we consider the more general (and less rigid) level sets C_A instead of $\widetilde{C}_A$. A good reason for this is the connection to nonlinear level sets in Section 3.1.

2.2. **Dimension theory and thermodynamic formalism.** We first recall a useful notion of dimension introduced in [8]. Let $T\colon X \to X$ be a continuous map of a compact metric space. Given a finite open cover $\mathcal{U}$ of X, for each $n \in \mathbb{N}$ let $\mathcal{X}_n$ be the set of strings $U = (U_1, \ldots, U_n)$ with $U_i \in \mathcal{U}$ for $i = 1, \ldots, n$. We write $m(U) = n$ and define

$$X(U) = \big\{x \in X : T^{k-1}(x) \in U_k \text{ for } k = 1, \ldots, m(U)\big\}.$$

We say that a set $\Gamma \subset \bigcup_{n\in\mathbb{N}} \mathcal{X}_n$ *covers* a set $Z \subset X$ if $Z \subset \bigcup_{U\in\Gamma} X(U)$. Given a continuous function $u : X \to \mathbb{R}^+$, for each $U \in \mathcal{X}_n$ we define

$$u(U) = \begin{cases} \sup_{X(U)} \frac{1}{n} S_n u(x) & \text{if } X(U) \neq \emptyset, \\ -\infty & \text{if } X(U) = \emptyset. \end{cases}$$

Finally, given a set $Z \subset X$ and a number $\alpha \in \mathbb{R}$, let

$$N_Z(\alpha, u, \mathcal{U}) = \lim_{n\to\infty} \inf_{\Gamma} \sum_{U\in\Gamma} \exp(-\alpha u(U)),$$

with the infimum taken over all $\Gamma \subset \bigcup_{k\geq n} \mathcal{X}_k$ covering Z and with the convention that $\exp(-\infty) = 0$. Denoting by $\operatorname{diam}\mathcal{U}$ the diameter of $\mathcal{U}$, one can show that

$$\dim_u Z := \lim_{\operatorname{diam}\mathcal{U}\to 0} \dim_{u,\mathcal{U}} Z, \quad \text{where } \dim_{u,\mathcal{U}} Z = \inf\bigl\{\alpha \in \mathbb{R} : N_Z(\alpha, u, \mathcal{U}) = 0\bigr\},$$

is well defined. This is called the *u-dimension* of Z with respect to T. When $u = 1$, we have $\dim_u Z = h_{\text{top}}(T|Z)$, the *topological entropy* of T restricted to Z (see [9]). Moreover, when X is a repeller of a C^1 conformal expanding map $T\colon X \to X$ and $u(x) = \log \|d_x T\|$, we have $\dim_u Z = \dim_H Z$ for every set $Z \subset X$, where $\dim_H Z$ denotes the *Hausdorff dimension* of Z. The function $\beta \mapsto h_{\text{top}}(T|C_\beta^F)$ is called the *nonlinear entropy spectrum* and the functions $\beta \mapsto \dim_u C_\beta^F$ and $\beta \mapsto \dim_H C_\beta^F$ are called *nonlinear dimension spectra*.

Given a Borel probability measure ν on X, the limit

$$\dim_u \nu = \lim_{\operatorname{diam}\mathcal{U}\to 0} \inf\bigl\{\dim_{u,\mathcal{U}} Z : \nu(Z) = 1\bigr\}$$

is called the *u-dimension of* ν. We also introduce local quantities that generalize the notion of pointwise dimensions. The *lower* and *upper u-pointwise dimensions of* ν at a point $x \in X$ are defined respectively by

$$\underline{d}_{\nu,u}(x) = \lim_{\operatorname{diam}\mathcal{U}\to 0} \liminf_{n\to\infty} \inf_{U} -\frac{\log \nu(X(U))}{u(U)},$$

$$\overline{d}_{\nu,u}(x) = \lim_{\operatorname{diam}\mathcal{U}\to 0} \limsup_{n\to\infty} \sup_{U} -\frac{\log \nu(X(U))}{u(U)}.$$

When $\nu \in \mathcal{M}(T)$ is ergodic, it was proved in [8] that

$$\dim_u \nu = \underline{d}_{\nu,u}(x) = \overline{d}_{\nu,u}(x) = \frac{h_\nu(T)}{\int_X u\, d\nu},$$

where $h_\nu(T)$ is the *Kolmogorov–Sinai entropy* of T with respect to $\nu \in \mathcal{M}(T)$.

Pesin and Pitskel' extended the notion of topological pressure to noncompact sets in [31]. Given a continuous function $\varphi\colon X \to \mathbb{R}$, a set $Z \subset X$ and a number $\alpha \in \mathbb{R}$, let

$$M_Z(\alpha, \varphi, \mathcal{U}) := \lim_{n\to\infty} \inf_{\Gamma} \sum_{U\in\Gamma} \exp(-\alpha m(U) + \varphi(U)),$$

with the infimum taken over all $\Gamma \subset \bigcup_{k\geq n} \mathcal{X}_k$ covering Z. Letting

$$P_Z(\varphi, \mathcal{U}) = \inf\bigl\{\alpha \in \mathbb{R} : M_Z(\alpha, \varphi, \mathcal{U}) = 0\bigr\},$$

one can show that the limit

$$P_Z(\varphi) = \lim_{\operatorname{diam}\mathcal{U}\to 0} P_Z(\varphi, \mathcal{U})$$

exists. It is called the *topological pressure* of φ on the set $Z \subset X$ (with respect to T). Note that $s = \dim_u Z$ is the unique solution of the equation $P_Z(-tu) = 0$. When Z is the whole space X, we also write $P_X(\varphi) = P(\varphi)$.

Finally, we recall the notion of nonlinear topological pressure considered in [12] (see also [4]). Given $n \in \mathbb{N}$ and $\varepsilon > 0$, a set $E \subset X$ is said to be (n, ε)-*separated* if $d_n(x, y) > \varepsilon$ for every $x, y \in E$ with $x \neq y$. Since X is compact, any (n, ε)-separated set is finite. Let $F\colon \mathbb{R}^d \to \mathbb{R}$ be a continuous function and let $\Phi = \{\phi_1, \dots, \phi_d\}$ be a family of continuous functions $\phi_i\colon X \to \mathbb{R}$ for $i = 1, \dots, d$. The *nonlinear topological pressure* of Φ with respect to F is defined by

$$P_F(\Phi) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \sup_E \sum_{x\in E} \exp\left[nF\left(\frac{S_n\Phi(x)}{n}\right)\right],$$

with the supremum taken over all (n, ε)-separated sets E.

Following [12], we say that the pair (T, Φ) has *an abundance of ergodic measures* if for each $\mu \in \mathcal{M}(T)$, $h < h_\mu(T)$ and $\varepsilon > 0$ there exists an ergodic measure $\nu \in \mathcal{M}(T)$ such that $h_\nu(T) > h$ and

$$\left|\int_X \phi_i \, d\nu - \int_X \phi_i \, d\mu\right| < \varepsilon \quad \text{for } i = 1, \dots, d.$$

When (T, Φ) has an abundance of ergodic measures or F is a convex function, we have the variational principle

$$P_F(\Phi) = \sup_\mu \left(h_\mu(T) + F\left(\int_X \Phi \, d\mu\right)\right),$$

with the supremum taken over measures $\mu \in \mathcal{M}(T)$. A measure $\eta \in \mathcal{M}(T)$ is said to be an *equilibrium measure for* (F, Φ) with respect to T if

$$P_F(\Phi) = h_\eta(T) + F\left(\int_X \Phi \, d\eta\right).$$

When $\Phi = \{\varphi\}$ and F is the identity function, we recover the classical topological pressure $P_X(\varphi)$ of φ with respect to T. A measure $\mu \in \mathcal{M}(T)$ is said to be an *equilibrium measure* for φ if

$$P_X(\varphi) = h_\mu(T) + \int_X \varphi \, d\mu.$$

3. Nonlinear dimension spectra

In this section we explore different types of nonlinear entropy and dimension spectra and we study some relations between them and the classical topological pressure.

3.1. **Nonlinear conditional variational principles.**

First we establish several nonlinear conditional variational principles for the entropy and dimension spectra using results coming solely from the thermodynamic formalism. In particular, this allow us to give more detailed information about the measures of maximal entropy and maximal dimension.

Let $C^0(X)$ be the set of continuous functions $f\colon X \to \mathbb{R}$ and let $\mathcal{M}_Z(T) \subset \mathcal{M}(T)$ be the set of measures $\mu \in \mathcal{M}(T)$ with $\mu(Z) = 1$. The following theorem can be obtained combining Theorem C, Theorem 3.3 and Proposition 2.14 in [13].

Theorem 2 ([13]). *Let $T\colon X \to X$ be a continuous map of a compact metric space such that the metric entropy $\mu \mapsto h_\mu(T)$ is upper semicontinuous and $h_{\text{top}}(T) < \infty$. Suppose that $u\colon X \to \mathbb{R}^+$ is a continuous function, $\Phi = \{\phi_1, \dots, \phi_d\}$ is a collection of real-valued continuous functions on X and that there exists a dense subspace $D(X) \subset C^0(X)$ such that every $\xi \in D(X)$ has a unique equilibrium measure. Then:*

1. *for each compact set* $A \subset \operatorname{int} L$ *we have*

$$\begin{aligned}\dim_u C_A &= \sup_{\alpha\in A}\sup_{\mu\in\mathcal{M}(T)}\left\{\frac{h_\mu(T)}{\int_X u\,d\mu} : \int_X \Phi\,d\mu = \alpha\right\}\\ &= \sup_{\alpha\in A}\sup_{\mu\in\mathcal{M}(T)}\left\{\frac{h_\mu(T)}{\int_X u\,d\mu} : \mu\in\mathcal{M}_{C_\alpha}(T)\right\}\\ &= \sup_{\alpha\in A}\inf_{q\in\mathbb{R}^d} S_u(\alpha,q) = \sup_{\alpha\in A}\dim_u C_\alpha,\end{aligned}$$

where $S_u(\alpha,q)$ *is the unique number such that* $P(\langle q,\Phi-\alpha\rangle - S_u(\alpha,q)u) = 0$;

2. *given* $\alpha\in\operatorname{int} L$, *for each* $\varepsilon>0$ *there exists an ergodic measure* $\nu\in\mathcal{M}(T)$ *with* $\int_X \Phi\,d\nu = \alpha$ *and* $\nu(C_\alpha)=1$ *such that*

$$\left|\dim_u \nu - \dim_u C_\alpha\right| < \varepsilon;$$

3. *the map* $\operatorname{int} L \ni \alpha \mapsto \dim_u C_\alpha$ *is continuous.*

The next theorem is our main result. It is obtained through a correspondence between level sets with different limit points and nonlinear level sets, combined with Theorem 2. As far as we know, this is the first relation between the topological pressure and the nonlinear conditional variational principle. Furthermore, we approximate the nonlinear dimension spectrum via ergodic measures, which is crucial for studying nontypical points in the nonlinear case (see Section 7), for establishing some connections to multiple ergodic averages (see Section 9), and for extending intermediate entropy and dimension properties with respect to some hyperbolic systems (see Section 10).

Theorem 3 (Nonlinear conditional variational principle)**.** *Let* $T\colon X\to X$ *be a continuous map of a compact metric space such that the metric entropy* $\mu\mapsto h_\mu(T)$ *is upper semicontinuous and* $h_{\text{top}}(T)<\infty$. *Suppose that* $u\colon X\to\mathbb{R}^+$ *is a continuous function,* $\Phi=\{\phi_1,\dots,\phi_d\}$ *is a collection of real-valued continuous functions on* X *and that there exists a dense subspace* $D(X)\subset C^0(X)$ *such that every* $\xi\in D(X)$ *has a unique equilibrium measure. Moreover, let* $F\colon U\to\mathbb{R}^p$ *be a continuous function, where* U *is an open set containing* L. *Then for each* $\beta\in\mathbb{R}^p$ *with* $F^{-1}\beta\cap L\subset\operatorname{int} L$ *we have:*

1.

$$\begin{aligned}\dim_u C_\beta^F &= \sup_{\mu\in\mathcal{M}(T)}\left\{\frac{h_\mu(T)}{\int_X u\,d\mu} : F\left(\int_X \Phi\,d\mu\right) = \beta\right\}\\ &= \sup_{\alpha\in F^{-1}\beta}\sup_{\mu\in\mathcal{M}(T)}\left\{\frac{h_\mu(T)}{\int_X u\,d\mu} : \int_X \Phi\,d\mu = \alpha\right\}\\ &= \sup_{\alpha\in F^{-1}\beta}\sup_{\mu\in\mathcal{M}(T)}\left\{\frac{h_\mu(T)}{\int_X u\,d\mu} : \mu\in\mathcal{M}_{C_\alpha}(T)\right\}\\ &= \sup_{\alpha\in F^{-1}\beta}\inf_{q\in\mathbb{R}^d} S_u(\alpha,q) = \sup_{\alpha\in F^{-1}\beta}\dim_u C_\alpha,\end{aligned} \tag{2}$$

where $S_u(\alpha,q)$ *is the unique number such that* $P(\langle q,\Phi-\alpha\rangle - S_u(\alpha,q)u) = 0$;

2. *given* $\varepsilon>0$, *there exists an ergodic measure* $\nu\in\mathcal{M}(T)$ *with* $F(\int_X \Phi\,d\nu)=\beta$ *and* $\nu(C_\beta^F)=1$ *such that*

$$\left|\dim_u \nu - \dim_u C_\beta^F\right| < \varepsilon.$$

Proof. We need an auxiliary key lemma, which connects level sets with distinct limit points and nonlinear level sets.

Lemma 1. *Let* $T\colon X\to X$ *be a continuous map of a compact metric space and let* $\Phi\colon X\to\mathbb{R}^d$ *be a continuous function. Given a continuous function* $F\colon\Phi(X)\subset\mathbb{R}^d\to\mathbb{R}^p$, *we have* $C_{F^{-1}\beta}=C_\beta^F$ *for any* $\beta\in\mathbb{R}^p$.

Proof of the lemma. Take $x \in C_\beta^F$. Then the sequence $(F(S_n\Phi(x)/n))_{n\in\mathbb{N}}$ converges to β. Given $\alpha \in S_\infty(x)$, there is a subsequence $(S_{n_k}\Phi(x)/n_k)_{k\in\mathbb{N}}$ converging to α. Since F is continuous, this readily implies that

$$F(\alpha) = F\left(\lim_{k\to\infty}\frac{1}{n_k}S_{n_k}\Phi(x)\right) = \lim_{k\to\infty} F\left(\frac{1}{n_k}S_{n_k}\Phi(x)\right) = \beta,$$

and so $\alpha \in F^{-1}\beta$. Therefore, $S_\infty(x) \subset F^{-1}\beta$ and we obtain that $x \in C_{F^{-1}\beta}$.

Now take $x \in C_{F^{-1}\beta}$. This means that $S_\infty(x) \subset F^{-1}\beta$, which readily implies that $F(S_\infty(x)) \subset \{\beta\}$. We also consider the set

$$S_\infty^F(x) = \left\{\gamma \in \mathbb{R}^p : \text{there exists a subsequence } F\left(\frac{1}{m_k}S_{m_k}\Phi(x)\right) \to \gamma\right\}.$$

We claim that $S_\infty^F(x) \subset F(S_\infty(x))$. In fact, consider a subsequence satisfying $F(\frac{1}{m_k}S_{m_k}\Phi(x)) \to \gamma$. Since $(S_n\Phi(x)/n)_{\in\mathbb{N}}$ is bounded, one can take a subsequence ℓ_k of m_k such that $\frac{1}{\ell_k}S_{\ell_k}\Phi(x)$ converges to some $\alpha \in \mathbb{R}^d$. Thus,

$$F(\alpha) = \lim_{k\to\infty} F\left(\frac{1}{\ell_k}S_{\ell_k}\Phi(x)\right) = \lim_{k\to\infty} F\left(\frac{1}{m_k}S_{m_k}\Phi(x)\right) = \gamma$$

and $\alpha \in S_\infty(x)$. This shows that $\gamma \in F(S_\infty(x))$, and finally we have $S_\infty^F(x) \subset F(S_\infty(x)) \subset \{\beta\}$. Therefore, the sequence $F(\frac{1}{n}S_n\Phi(x))_{n\in\mathbb{N}}$ has the unique limit point β. Moreover, since this sequence is bounded, it converges to β, that is, $x \in C_\beta^F$. ☐

Since each closed set $F^{-1}\beta \subset L$ is compact, all the equalities in (2) follow directly from the first item in Theorem 2 together with Lemma 1.

Now we prove item 2. By item 3 in Theorem 2, the dimension spectrum $\operatorname{int} L \ni \alpha \mapsto \dim_u C_\alpha$ is continuous. Since $F^{-1}\beta \subset \operatorname{int} L$ is compact, there exists $\alpha^* \in F^{-1}\beta$ such that

$$\dim_u C_\beta^F = \sup_{\alpha\in F^{-1}\beta} \dim_u C_\alpha = \dim_u C_{\alpha^*}.$$

It follows from item 2 in Theorem 2 that for each $\varepsilon > 0$ there exists an ergodic measure $\nu \in \mathcal{M}(T)$ concentrated on $C_{\alpha^*}(\Phi)$ with $\int_X \Phi\,d\nu = \alpha^*$ such that

$$\left|\dim_u C_{\alpha^*}(\Phi) - \dim_u \nu\right| < \varepsilon.$$

Since $C_{\alpha^*} \subset C_\beta^F$, we also have $\nu(C_\beta^F) = 1$. Moreover, $F(\int_X \Phi\,d\nu) = F(\alpha^*) = \beta$, and the theorem is proved. ☐

Taking the constant function $u = 1$ in Theorem 3, we obtain a corresponding result for the nonlinear entropy spectrum. Furthermore, in the case of conformal repellers one can apply Theorem 3 to unify many results in the literature for entropy and dimension spectra. This includes for example Theorems 3.4, 3.5 and 3.6 in [13]. See Section 3.3 for further applications.

We also observe that any level set can actually be considered as a particular type of a nonlinear level set.

Proposition 4. *Let $T\colon X \to X$ be a continuous map of a compact metric space and let $\Phi\colon X \to \mathbb{R}^d$ be a continuous function. Given a compact set $A \subset L$ and a number $\beta \in \mathbb{R}$, there exists a C^∞ function $F\colon \mathbb{R}^d \to \mathbb{R}$ such that $C_A = C_\beta^F$.*

Proof. It is well known that any compact set $A \subset \mathbb{R}^d$ can be realized as the set of solutions of an equation $G(x) = 0$, where $G\colon \mathbb{R}^d \to [0,\infty)$ is a C^∞ function. Thus, $F^{-1}\beta = A$ for the function $F(x) = G(x) + \beta$, and so $C_A = C_{F^{-1}\beta}$. The conclusion now follows from Lemma 1. ☐

By Lemma 1 and Proposition 4, each set C_A is equal to $C_{G^{-1}\lambda} = C^G_\lambda$ for some smooth function $G\colon \mathbb{R}^d \to \mathbb{R}$ and some $\lambda \in \mathbb{R}$. Thus, in the nonlinear conditional variational principle in Theorem 3, it suffices to consider real-valued C^∞ functions F.

Furthermore, again as a consequence of Lemma 1 and Proposition 4, one can obtain a corresponding result for general level sets containing divergent points as defined in (1), under the setup of Theorem 7.2 in [36].

Corollary 5. *Let $T\colon X \to X$ be a continuous map of a compact metric space satisfying the specification property and such that the entropy map $\mu \mapsto h_\mu(T)$ is upper semicontinuous. Suppose that $\Phi\colon X \to \mathbb{R}^d$ is a continuous function. Then for each compact set $A \subset \mathbb{R}^d$ with $C_A \neq \emptyset$ we have*

$$h_{\text{top}}(T|C_A) = \sup_{\alpha\in A} \sup_{\mu\in\mathcal{M}(T)} \left\{ h_\mu(T) : \int_X \Phi \, d\mu = \alpha \right\} = \sup_{\alpha \in A} h_{\text{top}}(T|C_\alpha).$$

Remark 1. Similarly, Lemma 1 and Proposition 4 allow us to obtain correspondences between many results involving nonlinear conditional variational principles and conditional variational principles for level sets containing divergent points presented in [27] and [28]. We refrain from detailing these applications explicitly.

We continue to assume that $T\colon X \to X$ is a continuous map of a compact metric space, $u\colon X \to \mathbb{R}$ is a strictly positive continuous function, and $\Phi\colon X \to \mathbb{R}^d$ and $F\colon \mathbb{R}^d \to \mathbb{R}^p$ are continuous functions. Given $\alpha \in L$, a measure $\mu \in \mathcal{M}(T)$ is called a *full measure for the level set* C_α or simply a *full measure for* α if

$$\int_X \Phi \, d\mu = \alpha, \quad \mu(C_\alpha) = 1 \quad \text{and} \quad \dim_u C_\alpha = \frac{h_\mu(T)}{\int_X u \, d\mu}.$$

Then we also say that α *admits a full measure*. Similarly, $\nu \in \mathcal{M}(T)$ is called a *full measure for the nonlinear level set* C^F_β or simply a *nonlinear full measure for* β *(with respect to* F*)* if

$$F\left(\int_X \Phi \, d\nu\right) = \beta, \quad \nu(C^F_\beta) = 1 \quad \text{and} \quad \dim_u C^F_\beta = \frac{h_\nu(T)}{\int_X u \, d\nu}.$$

Then we also say that β *admits a nonlinear full measure (with respect to* F*)*.

Inspired by the multifractal formalisms developed in [5] and [7], we also obtain a sharper nonlinear conditional variational principle with all the suprema in Theorem 3 replaced by maxima under more restrictive assumptions. Furthermore, we also can guarantee the existence of ergodic nonlinear full measures. Let $U_E(X) \subset C^0(X)$ be the set of continuous functions having a unique equilibrium measure.

Theorem 6. *Let $T\colon X \to X$ be a continuous map of a compact metric space such that the metric entropy $\mu \mapsto h_\mu(T)$ is upper semicontinuous and $h_{\text{top}}(T) < \infty$. Suppose that $u\colon X \to \mathbb{R}^+$ is a continuous function and that $\Phi = \{\phi_1, \ldots, \phi_d\}$ is a collection of real-valued continuous functions on X such that* $\operatorname{span}\{u, \phi_1, \ldots, \phi_d\} \subset U_E(X)$. *Moreover, let $F\colon U \to \mathbb{R}^p$ be a continuous function on some open set U containing L. Then for each $\beta \in \mathbb{R}^p$ with $F^{-1}\beta \cap L \subset \operatorname{int} L$ we have:*

1.

$$
\begin{aligned}
\dim_u C_\beta^F &= \max_{\mu\in\mathcal{M}(T)}\left\{\frac{h_\mu(T)}{\int_X u\,d\mu} : F\left(\int_X \Phi\,d\mu\right)=\beta\right\}\\
&= \max_{\alpha\in F^{-1}\beta}\max_{\mu\in\mathcal{M}(T)}\left\{\frac{h_\mu(T)}{\int_X u\,d\mu} : \int_X \Phi\,d\mu=\alpha\right\}\\
&= \max_{\alpha\in F^{-1}\beta}\max_{\mu\in\mathcal{M}(T)}\left\{\frac{h_\mu(T)}{\int_X u\,d\mu} : \mu\in\mathcal{M}_{C_\alpha}(T)\right\}\\
&= \max_{\alpha\in F^{-1}\beta}\min_{q\in\mathbb{R}^d} S_u(\alpha,q) = \max_{\alpha\in F^{-1}\beta}\dim_u C_\alpha,
\end{aligned}
\tag{3}
$$

where $S_u(\alpha,q)$ *is the unique number such that* $P(\langle q,\Phi-\alpha\rangle - S_u(\alpha,q)u)=0$*;*

2. *there exists an ergodic nonlinear full measure* ν_β *for* β*, thus satisfying*

$$\dim_u C_\beta^F = \dim_u \nu_\beta.$$

Proof. The proof of this theorem uses some important elements from the classical thermodynamic formalism, of which the main ingredient is the following result (see Theorem 4.2.11 in [25]).

Lemma 2. *Let* $T\colon X\to X$ *be a continuous map of a compact metric space such that the metric entropy* $\mu\mapsto h_\mu(T)$ *is upper semicontinuous. Then:*

- *given* $\varphi\in C^0(X)$*, the map* $\mathbb{R}\ni t\mapsto P(\varphi+t\psi)$ *is differentiable at* $t=0$ *for all* $\psi\in C^0(X)$ *if and only if* $\varphi\in U_E(X)$*, in which case the unique equilibrium measure* η_φ *for* φ *is ergodic and*

$$\frac{d}{dt}P(\phi+t\psi)|_{t=0} = \int_X \psi\ d\eta_\phi;$$

- *if* $\varphi,\psi\in C^0(X)$ *are such that* $\operatorname{span}\{\varphi,\psi\}\subset U_E(X)$*, then the function* $t\mapsto P(\varphi+t\psi)$ *is of class* C^1*.*

We proceed with the proof of the theorem, which is a combination of Lemma 1 and Theorem 2 together with Theorem 8 in [7]. Given $\alpha\in L$, we consider the function $\Gamma_\alpha\colon\mathbb{R}^d\to\mathbb{R}$ defined by

$$\Gamma_\alpha(q) = P\big(\langle q,\Phi-\alpha\rangle - \dim_u C_\alpha u\big).$$

By Lemmas 1 and 2 in [7] we have

$$\inf_{q\in\mathbb{R}^d}\Gamma_\alpha(q)\ge 0 \quad \text{for } \alpha\in L,$$

$$\inf_{q\in\mathbb{R}^d}\Gamma_\alpha(q)= 0 \quad \text{for } \alpha\in \operatorname{int} L,$$

and there exists $q(\alpha)\in\mathbb{R}^d$ such that $\Gamma_\alpha(q(\alpha))=0$. Since $\operatorname{span}\{u,\phi_1,...,\phi_d\}\subset U_E(X)$, by Lemma 2 the map $q\mapsto\Gamma_\alpha(q)$ is of class C^1, and so $\partial_q\Gamma_\alpha(q(\alpha))=0$. Now let μ_α be the unique equilibrium measure of the potential

$$\mathcal{H}_\alpha = \langle q(\alpha),\Phi-\alpha\rangle - \dim_u C_\alpha u. \tag{4}$$

By Lemma 2 we have

$$\int_X(\Phi-\alpha)\,d\mu_\alpha = \partial_q\Gamma_\alpha(q(\alpha))=0.$$

Proceeding as in the proof of Theorem 8 in [7], one also can verify that μ_α is ergodic and satisfies

$$\mu_\alpha(C_\alpha)=1 \quad\text{and}\quad \dim_u C_\alpha = \frac{h_{\mu_\alpha(T)}}{\int_X u\,d\mu_\alpha} = \dim_u\mu_\alpha. \tag{5}$$

Moreover, since $\min_{q\in\mathbb{R}^d}\Gamma_\alpha(q)=\Gamma_\alpha(q(\alpha))$, using the definition of $S_u(\alpha,q)$ one can verify that the spectrum is given by

$$\dim_u C_\alpha=\min_{q\in\mathbb{R}^d} S_u(\alpha,q)=S_u(\alpha,q(\alpha)).$$

Since $F^{-1}\beta\subset\operatorname{int}L$ is compact and the dimension spectrum $\alpha\mapsto\dim_u C_\alpha$ is continuous on $\operatorname{int}L$, by Lemma 1 and Theorem 2 we finally obtain

$$\dim_u C_\beta^F=\dim_u C_{F^{-1}\beta}=\max_{\alpha\in F^{-1}\beta}\dim_u C_\alpha.$$

This establishes all the identities in (3).

Let us now establish the second item. Observe that there exists $\alpha^*\in F^{-1}\beta$ such that $\dim_u C_\beta^F=\dim_u C_{\alpha^*}$ and $C_{\alpha*}\subset C_\beta^F$. Together with (5) this yields

$$\mu_{\alpha^*}(C_\beta^F)=1,\quad F\left(\int_X\Phi\,d\mu_{\alpha^*}\right)=\beta\quad\text{and}\quad\dim_u C_\beta^F=\dim_u\mu_{\alpha^*}.$$

Hence, the result follows by taking $\nu_\beta=\mu_{\alpha^*}$, which completes the proof of the theorem. □

Remark 2. Besides being useful for applications, Theorem 6 also gives a sufficiently general setup that allow one to extract the optimal outcome of the nonlinear multifractal analysis via the thermodynamic formalism (in the sense that one can guarantee the existence of nonlinear full measures). In addition, it allows one to recover several related results in the literature; In particular, if we consider the nonlinear function $F\colon\mathbb{R}^{2d}\to\mathbb{R}^d$ given by

$$F(x_1,\dots,x_d,x_{d+1},\dots,x_{2d})=\left(\frac{x_1}{x_{d+1}},\dots,\frac{x_d}{x_{2d}}\right)$$

and the assumption that $\phi_j>0$ for all $d+1\le j\le 2d$, we recover Theorem 8 in [7]. As we shall see in the next section, Theorems 3 and 6 extend and unify many results in the literature, such as in [7, 8, 13, 14], for various mixed and nonmixed entropy and dimension spectra in one and higher-dimensions.

3.2. **Further developments.** Before proceeding, we briefly recall some notions that are necessary in this section. Given $\delta>0$, we say that T has *weak specification at scale* δ if there exists $\tau\in\mathbb{N}$ such that for any pairs $(x_1,n_1),\dots,(x_k,n_k)\in X\times\mathbb{N}$ there are $y\in X$ and $\tau_1,\dots,\tau_{k-1}\in\mathbb{N}$ such that $\tau_i\le\tau$ and

$$d_{n_i}(T^{s_{i-1}+\tau_{i-1}}(y),x_i)<\delta\quad\text{for } i=1,\dots,k,$$

where $s_i=\sum_{j=1}^i n_j+\sum_{j=1}^{i-1}\tau_j$ with $n_0=\tau_0=0$. When one can take $\tau_j=\tau$ for $j=1,\dots,k-1$, we say that T has *specification at scale* δ. Finally, we say that T has *weak specification* (respectively *specification*) if it has weak specification (respectively specification) at scale δ for all $\delta>0$.

A nonlinear conditional variational principle for the topological entropy was obtained by Takens and Verbitskiy in [36] for systems with the specification property and with an upper semicontinuous metric entropy. Their result was obtained by the so-called orbit-gluing approach, in particular showing no connection with the topological pressure nor presenting any information about measures concentrated on the level sets. They also asked whether the identity

$$h_{\text{top}}(T|C_\beta^F)=\sup_{\alpha\in F^{-1}\beta}h_{\text{top}}(T|C_\alpha)\tag{6}$$

holds for systems without the specification property.

Following [8], we give examples of dynamical systems without specification to which Theorems 3 and 6 still apply. Denote by $[x]$ and $\{x\}$ the integer and fractional parts of $x\in\mathbb{R}$, respectively. Let $\beta>1$ be a real number and define a map

$T\colon [0,1] \to [0,1]$ by $T(x) = \{\beta x\}$. The expansion of $x \in [0,1]$ in base β is the sequence of integers $(i_n)_{n\in\mathbb{N}}$ in $\{0,\dots,[\beta]\}$ defined by $i_n = [T^{n-1}(x)]$ for $n \in \mathbb{N}$. We endow the set $\Sigma_\beta = \{0,\dots,[\beta]\}^{\mathbb{N}}$ with the product topology and consider the shift map $\sigma\colon \Sigma_\beta \to \Sigma_\beta$ given by $\sigma(i_1 i_2 \cdots) = (i_2 i_3 \cdots)$, which is called the *$\beta$-shift*. We recall that any β-shift is expansive and has an upper semicontinuous metric entropy (see for example [25]). However, for β in a residual set of full Lebesgue measure, the corresponding β-shift does not satisfy the specification property (see [35]). It follows from [38] that each Lipschitz function has a unique equilibrium measure with respect to any β-shift, and so our Theorem 3 applies in this setting. Moreover, it yields a more general identity for the u-dimension, namely

$$\dim_u C_\beta^F = \sup_{\alpha\in F^{-1}\beta} \dim_u C_\alpha, \tag{7}$$

when $F^{-1}\beta \subset \operatorname{int} L$. In particular, (6) holds for any β-shift with β in a residual set.

We observe that when $T\colon X \to X$ does not satisfy the specification property and $F^{-1}\beta \cap \partial L \neq \emptyset$, it may happen that

$$\sup_{\alpha\in F^{-1}\beta} \sup_{\mu\in\mathcal{M}(T)} \left\{ \frac{h_\mu(T)}{\int_X u\,d\mu} : \int_X \Phi\,d\mu = \alpha \right\} > \sup_{\alpha\in F^{-1}\beta} \dim_u C_\alpha \tag{8}$$

(see the discussion in Section 3.4 of [14]). Now assume that T and Φ satisfy the hypotheses of Theorem 3 and that the spectrum $\alpha \mapsto \dim_u C_\alpha$ restricted to $F^{-1}\beta$ attains its maximum at some point $\alpha^* \in \operatorname{int} L$. Then it follows from Lemma 1, Theorem 3 and (8) that

$$\begin{aligned}
\dim_u C_\beta^F &= \dim_u C_{F^{-1}\beta} \ge \dim_u C_{\alpha^*} \\
&= \sup_{\mu\in\mathcal{M}(T)} \left\{ \frac{h_\mu(T)}{\int_X u\,d\mu} : F\left(\int_X \Phi\,d\mu\right) = \beta \right\} \\
&> \sup_{\alpha\in F^{-1}\beta} \dim_u C_\alpha,
\end{aligned}$$

On the other hand, when α^* is a maximizing point for the spectrum in the whole domain L we get

$$\dim_u C_\beta^F = \dim_u C_{F^{-1}\beta} \ge \dim_u C_{\alpha^*} = \dim_u X,$$

which yields identity (7), that is,

$$\dim_u C_\beta^F = \dim_u X = \sup_{\alpha\in F^{-1}\beta} \dim_u C_\alpha.$$

See Section 6 for a related discussion on the role of the maximizing points for the dimension spectrum.

3.3. **Nonlinear mixed spectra for repellers.** In this section we consider briefly the particular case of mixed spectra for conformal repellers. For simplicity of the exposition we assume from the beginning that $T\colon M \to M$ is a C^1 map of a Riemmanian manifold and that μ is a T-invariant Borel probability measure on M.

Let $B(x,r) \subset M$ be the open ball of radius r centered at x. We define the *pointwise dimension* of μ at $x \in M$ by

$$d_\mu(x) = \lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}$$

whenever the limit exists. Moreover, we define the *upper and lower μ-local entropies* of T at $x \in M$ respectively by

$$\overline{h}_\mu(x) = \lim_{\delta\to 0} \limsup_{n\to\infty} -\frac{1}{n} \log \mu(B_n(x,\delta)),$$

$$\underline{h}_\mu(x) = \lim_{\delta\to 0} \liminf_{n\to\infty} -\frac{1}{n} \log \mu(B_n(x,\delta)),$$

where $B_n(x,\delta)$ is the Bowen ball

$$B_n(x,\delta)=\big\{y\in M : d(T^k(x),T^k(y))<\varepsilon \text{ for all } 0\le k<n\big\}.$$

It was shown by Brin and Katok in [11] that $h_\mu(x):=\overline{h}_\mu(x)=\underline{h}_\mu(x)$ for μ-almost every $x\in M$. Finally, given $x\in M$, we define

$$\lambda(x)=\lim_{n\to\infty}\frac{1}{n}\log\|d_xT^n\|$$

whenever the limit exists. By Kingman's subadditive ergodic theorem, λ is well defined μ-almost everywhere.

Now let $J\subset M$ be a compact T-invariant set (that is, $T^{-1}J=J$). We say that T is *uniformly expanding* on J and that J is a repeller of T if there exist constants $c>0$ and $\lambda>1$ such that

$$\|d_xT^nv\|\ge c\lambda^n\|v\| \quad \text{for all } x\in J,\ n\in\mathbb{N} \text{ and } v\in T_xM.$$

Moreover, we say that T is *conformal* on J if d_xT is a multiple of an isometry for every $x\in J$.

In this section we denote by $\mathcal{M}$ the set of T-invariant Borel probability measures on J and by $\mathcal{M}_{\rm erg}\subset\mathcal{M}$ the subset of ergodic measures. Let $\Phi=\{\phi_1,\dots,\phi_d\}$ be a collection of real-valued Hölder continuous functions on J and define the sets

$$\mathsf{E}=\left\{-\int_J\Phi\,d\mu:\mu\in\mathcal{M}\right\}\subset\mathbb{R}^d,\quad \mathsf{L}=\left\{\int_J\log\|dT\|\,d\mu:\mu\in\mathcal{M}\right\}\subset\mathbb{R},$$

$$\mathsf{D}=\left\{-\frac{\int_J\Phi\,d\mu}{\int_J\log\|dT\|\,d\mu}:\mu\in\mathcal{M}\right\}\subset\mathbb{R}^d.$$

We say that a measure μ on M is a *weak Gibbs measure* for a continuous function $\phi\colon M\to\mathbb{R}$ (with respect to T) if for any sufficiently small $\delta>0$ there exists a sequence $(K_n(\delta))_{n\in\mathbb{N}}$ in $[1,\infty)$ with $\lim_{n\to\infty}\frac{1}{n}\log K_n(\delta)=0$ such that

$$K_n(\delta)^{-1}\le\frac{\mu(B_n(x,\delta))}{\exp[-nP(\phi)+S_n\phi(x)]}\le K_n(\delta)$$

for all $x\in M$ and $n\in\mathbb{N}$. Let ν_j be a weak Gibbs measure for each function ϕ_j in the collection Φ. We also consider the nonlinear level sets of:

- local entropies and Lyapunov exponents
$$EL^F_\tau=\big\{x\in J : F(h_{\nu_1}(x),\dots,h_{\nu_d}(x),\lambda(x))=\tau\big\};$$
- pointwise dimensions and Lyapunov exponents
$$DL^G_\tau=\big\{x\in J : G(d_{\nu_1}(x),\dots,d_{\nu_d}(x),\lambda(x))=\tau\big\};$$
- local entropies and pointwise dimensions
$$ED^H_\tau=\big\{x\in J : H(h_{\nu_1}(x),\dots,h_{\nu_d}(x),d_{\nu_1}(x),\dots,d_{\nu_d}(x))=\tau\big\},$$

where $F\colon U\subset\mathbb{R}^{d+1}\to\mathbb{R}^{d+1}$, $G\colon V\subset\mathbb{R}^{d+1}\to\mathbb{R}^{d+1}$ and $H\colon W\subset\mathbb{R}^{2d}\to\mathbb{R}^{2d}$ are continuous functions defined respectively on open sets $U\supset\mathsf{E}\times\mathsf{L}$, $V\supset\mathsf{D}\times\mathsf{L}$ and $W\supset\mathsf{E}\times\mathsf{D}$. Theorem 7 in [7] gives sufficient conditions so that the interior of the domains $\mathsf{E}\times\mathsf{L}$, $\mathsf{D}\times\mathsf{L}$ and $\mathsf{E}\times\mathsf{D}$ is nonempty.

Based on Theorem 3, we obtain a complete *nonlinear* multifractal analysis for the mixed and nonmixed higher-dimensional spectra.

Theorem 7. *Let J be a conformal repeller of a topologically mixing C^1 expanding map T. Suppose that all the functions in Φ have zero topological pressure and that*

ν_j is a weak Gibbs measure for ϕ_j for $j = 1, \dots, d$. Then taking $u(x) = \log \|d_x T\|$, for each $\tau \in \mathbb{R}^{d+1}$ with $F^{-1}\tau \cap (\mathsf{E} \times \mathsf{L}) \subset \operatorname{int}(\mathsf{E} \times \mathsf{L})$ we have

$$
\begin{aligned}
h_{\text{top}}(T|EL_\tau^F) &= \sup_{\mu \in \mathcal{M}_{\text{erg}}} \left\{ h_\mu(T) : F\left(-\int_J \Phi \, d\mu, \int_J u \, d\mu \right) = \tau \right\}, \\
\dim_H EL_\tau^F &= \sup_{\mu \in \mathcal{M}_{\text{erg}}} \left\{ \frac{h_\mu(T)}{\int_J u \, d\mu} : F\left(-\int_J \Phi \, d\mu, \int_J u \, d\mu \right) = \tau \right\}.
\end{aligned} \tag{9}
$$

As a consequence, we also obtain the following variational principles:

1. *for each $\tau \in \mathbb{R}^{d+1}$ with $G^{-1}\tau \cap (\mathsf{D} \times \mathsf{L}) \subset \operatorname{int}(\mathsf{D} \times \mathsf{L})$ we have*

$$
\begin{aligned}
h_{\text{top}}(T|DL_\tau^G) &= \sup_{\mu \in \mathcal{M}_{\text{erg}}} \left\{ h_\mu(T) : G\left(-\frac{\int_J \Phi \, d\mu}{\int_J u \, d\mu}, \int_J u \, d\mu \right) = \tau \right\}, \\
\dim_H DL_\tau^G &= \sup_{\mu \in \mathcal{M}_{\text{erg}}} \left\{ \frac{h_\mu(T)}{\int_J u \, d\mu} : G\left(-\frac{\int_J \Phi \, d\mu}{\int_J u \, d\mu}, \int_J u \, d\mu \right) = \tau \right\};
\end{aligned}
$$

2. *when $\Phi = \{\varphi\}$, for each $(\tau_1, \tau_2) \in \mathbb{R}^2$ with*

$$H^{-1}(\tau_1, \tau_2) \cap (\mathsf{E} \times \mathsf{D}) \subset \operatorname{int}(\mathsf{E} \times \mathsf{D})$$

we have

$$
\begin{aligned}
h_{\text{top}}(T|ED_\tau^H) &= \sup_{\mu \in \mathcal{M}_{\text{erg}}} \left\{ h_\mu(T) : H\left(-\int_J \varphi \, d\mu, -\frac{\int_J \varphi \, d\mu}{\int_J u \, d\mu} \right) = (\tau_1, \tau_2) \right\}, \\
\dim_H ED_\tau^H &= \sup_{\mu \in \mathcal{M}_{\text{erg}}} \left\{ \frac{h_\mu(T)}{\int_J u \, d\mu} : H\left(-\int_J \varphi \, d\mu, -\frac{\int_J \varphi \, d\mu}{\int_J u \, d\mu} \right) = (\tau_1, \tau_2) \right\}.
\end{aligned}
$$

Proof. Since T is differentiable and conformal on J, we have

$$\lambda(x) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \log \|d_{T^k(x)} T\| = \lim_{n \to \infty} \frac{S_n u(x)}{n}. \tag{10}$$

In addition, since each ν_j is a weak Gibbs measure with respect to ϕ_j and $P(\phi_j) = 0$, it follows readily that

$$h_{\nu_j}(x) = \lim_{n \to \infty} -\frac{S_n \phi_j(x)}{n} \quad \text{and} \quad d_{\nu_j}(x) = \lim_{n \to \infty} -\frac{S_n \phi_j(x)}{S_n u(x)} \tag{11}$$

for $j = 1, \dots, d$. Finally, it is well known that for a topologically mixing expanding map, every Hölder continuous function has a unique equilibrium measure. Since the set of Hölder continuous functions is dense in $C^0(X)$, one can apply Theorem 3 to obtain the nonlinear conditional variational principles for EL_τ^F in (9).

Now consider the function $Q \colon \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^{d+1}$ given by $Q(x, y) = (x/y, y)$. It follows from (10) and (11) that $DL_\tau^G = EL_\tau^{G \circ Q}$, which proves item 1. Finally, consider the function $R \colon \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^2$ given by $R(x, y) = (x, x/y)$. It follows again from (10) and (11) that $ED_\tau^H = EL_\tau^{H \circ R}$, which gives item 2. □

Remark 3. Note that we only considered the case of a single potential in the last item in Theorem 7. This is unavoidable since for $d > 1$ we always have $\operatorname{int}(\mathsf{E} \times \mathsf{D}) = \emptyset$ (see Section 4 in [7]).

When T is a topologically mixing $C^{1+\gamma}$ expanding map (for some $\gamma > 0$), each function in Φ is Hölder continuous, and each ν_j is the equilibrium measure for ϕ_j, one can use Theorem 6 to obtain sharper conditional variational principles, where each supremum over $\mathcal{M}_{\text{erg}}$ in Theorem 7 is replaced by a maximum. Under these hypotheses, Theorem 7 gives a nonlinear generalization of Theorem 6 in [7].

We also observe that one can use Theorems 3 and 6 to combine the aforementioned entropy and dimension spectra in various different ways, leading to nonlinear extensions of many results in the literature, including in particular of Theorems 3.8 and 3.10 in [13]. It is also worth mentioning that one can obtain nonlinear multifractal conditional variational principles for nonuniformly expanding conformal repellers making the appropriate technical modifications following [13].

4. Number and characterization of full measures

Theorem 8 in [7] and Theorem 6 give a setup where each $\alpha \in \operatorname{int} L$ admits a full measure, and where each $\beta \in \mathbb{R}^p$ satisfying $F^{-1}\beta \cap L \subset \operatorname{int} L$ admits a nonlinear full measure with respect to F. We are left with some natural related questions:

Q1. *Under the conditions of Theorem 8 in* [7]*, how many full measures are there for each $\alpha \in \operatorname{int} L$? And what type of measures are they?*

Q2. *Under the conditions of Theorem 6, how many nonlinear full measures are there for each β satisfying $F^{-1}\beta \cap L \subset \operatorname{int} L$? And does the number of full measures depend on the parameter β, on the nonlinear function F, or on the function u?*

We observe that **Q1** can be seen as a folklore-type question and was mentioned before in the literature (see [2] and [3]). Nevertheless, to the best of our knowledge, it was never systematically addressed in any setup.

We claim that each $\alpha \in \operatorname{int} L$ admits a unique full measure. Indeed, suppose that $\alpha \in \operatorname{int} L$ admits a full measure $\mu \in \mathcal{M}(T)$. In particular,

$$\dim_u C_\alpha = \frac{h_\mu(T)}{\int_X u\,d\mu} \quad \text{and} \quad \int_X \Phi\,d\mu = \alpha. \tag{12}$$

By Theorem 8 in [7], α admits a full measure which is also the unique equilibrium measure η_α for the potential $\mathcal{H}_\alpha$ in (4). It follows from (12) that

$$\begin{aligned} h_\mu(T) + \int_X \mathcal{H}_\alpha\,d\mu &= h_\mu(T) + \int_X \langle q(\alpha), \Phi - \alpha\rangle\,d\mu - \dim_u C_\alpha \int_X u\,d\mu \\ &= \underbrace{\left(\frac{h_\mu(T)}{\int_X u\,d\mu} - \dim_u C_\alpha\right)}_{=0} \int_X u d\mu = 0 \\ &= \underbrace{\left(\frac{h_{\eta_\alpha}(T)}{\int_X u\,d\eta_\alpha} - \dim_u C_\alpha\right)}_{=0} \int_X u\,d\eta_\alpha \\ &= h_{\eta_\alpha}(T) + \int_X \mathcal{H}_\alpha\,d\eta_\alpha = P(\mathcal{H}_\alpha). \end{aligned}$$

Hence, μ is also an equilibrium measure for $\mathcal{H}_\alpha$, which readily implies that $\mu = \eta_\alpha$. This shows that the full measures are ergodic and unique. Note that when $\alpha_1 \neq \alpha_2$, the full measure ν for α_1 is different from the full measure η for α_2. Indeed, if $\mu = \nu = \eta$, we would have $\mu(C_{\alpha_1}) = \mu(C_{\alpha_2}) = 1$, and so also $\mu(C_{\alpha_1} \cap C_{\alpha_2}) = 1$. However, since $\alpha_1 \neq \alpha_2$, we have $C_{\alpha_1} \cap C_{\alpha_2} = \emptyset$.

Now we consider question **Q2**. By the proof of Theorem 6, each nonlinear full measure for β is also the unique equilibrium measure for some potential $\mathcal{H}_\alpha$ with $F(\alpha) = \beta$. Moreover, the number of nonlinear full measures for β is the number of points $\alpha^* \in F^{-1}\beta$ maximizing the dimension spectrum $\alpha \mapsto \dim_u C_\alpha(\Phi)$. In this case, a very rough upper bound for the quantity of nonlinear full measures for β is $\#F^{-1}\beta$.

Proposition 8. *Under the hypotheses of Theorem 6, the following holds:*

- **A.** *If* $F\colon \mathbb{R}^d \to \mathbb{R}^d$ *is invertible, then each* β *admits a unique nonlinear full measure.*
- **B.** *When* $\Phi = \{\phi\}$*, if the dimension spectrum* $\alpha \mapsto \dim_u C_\alpha$ *is strictly concave on* $\operatorname{int} L$*, then each* β *admits at most two nonlinear full measures.*
- **C.** *When* $\Phi = \{\phi\}$*, if the dimension spectrum* $\alpha \mapsto \dim_u C_\alpha$ *is analytic on* $\operatorname{int} L$*, then each* β *admits finitely many nonlinear full measures.*

Proof. Item **A** is immediate since $\#F^{-1}\beta = 1$ for all β with $F^{-1}\beta \cap L \subset \operatorname{int} L$. For item **B**, we observe that if some β with $F^{-1}\beta \cap L \subset \operatorname{int} L$ admits three or more nonlinear full measures, then the dimension spectrum attains a maximum at three or more different elements of $F^{-1}\beta$, and so it cannot be strictly concave on the open interval $\operatorname{int} L$.

Now we prove item **C**. It is well known that if an analytic function has infinitely many zeros on a compact interval, then it is identically zero. This implies that any nonconstant analytic function on the compact interval L has at most finitely many critical points. Since the dimension spectrum $\alpha \mapsto \dim_u C_\alpha$ is assumed to be analytic on $\operatorname{int} L$, it follows that

$$\#\{\text{nonlinear full measures for } \beta\} \le \#\{\text{critical points of } \dim_u C_\alpha | \operatorname{int} L\} + 1$$

for each β with $F^{-1}\beta \cap L \subset \operatorname{int} L$, and the desired result follows. □

We emphasize that all cases **A**, **B** and **C** in Proposition 8 occur naturally in the literature (see Sections 5 and 6).

When the collection Φ has more than one potential, and even when the dimension spectrum is strictly concave and analytic, one may have parameters β admitting uncountably many nonlinear full measures, as the following example shows.

Example 3. (Uncountably many nonlinear full measures). Let $\Sigma = \{1, 2, 3\}^{\mathbb{N}}$ and consider the shift map $\sigma\colon \Sigma \to \Sigma$ and the collection $\Phi = \{\phi_1, \phi_2\}$ composed of the functions $\phi_1, \phi_2\colon \Sigma \to \mathbb{R}$ given by

$$\phi_1 = 1_{C_1} \quad \text{and} \quad \phi_2 = 1_{C_3},$$

where 1_A is the indicator function of the set A, and C_j is the cylinder set of all sequences $\omega = (\omega_0\omega_1 \dots) \in \Sigma$ with $\omega_0 = j$. In this case,

$$L = \big\{(\mu(C_1), \mu(C_3)) : \mu \in \mathcal{M}(\sigma)\big\} \subset [0,1] \times [0,1].$$

Since $\mu(C_1) + \mu(C_2) + \mu(C_3) = 1$ for every measure $\mu \in \mathcal{M}(\sigma)$, Theorem 6 gives

$$\begin{aligned}\mathcal{E}(\alpha_1, \alpha_2) := h_{\text{top}}(\sigma | C_{(\alpha_1,\alpha_2)}) &= \max_{\mu \in \mathcal{M}(\sigma)} \big\{h_\mu(\sigma) : (\mu(C_1), \mu(C_3)) = (\alpha_1, \alpha_2)\big\} \\ &= -\alpha_1 \log \alpha_1 - \alpha_2 \log \alpha_2 - (1 - \alpha_1 - \alpha_2) \log(1 - \alpha_1 - \alpha_2).\end{aligned}$$

Moreover, clearly $L = \big\{(\alpha_1, \alpha_2) \in [0,1] \times [0,1] : \alpha_1 + \alpha_2 \le 1\big\}$. Notice that $\operatorname{int} L \ne \emptyset$ and that the spectrum $(\alpha_1, \alpha_2) \mapsto \mathcal{E}(\alpha_1, \alpha_2)$ is strictly concave on L (see Figure 2).

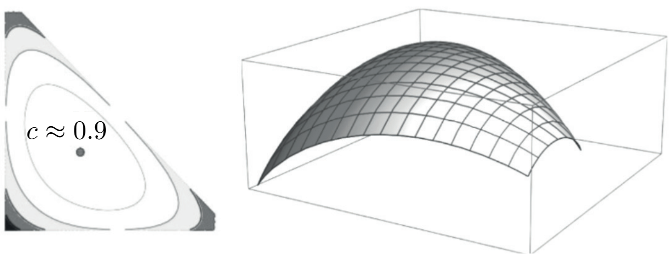

Figure 2. The domain L and the entropy spectrum $\mathcal{E}\colon L \to \mathbb{R}$.

Fix a number $c \in [0, \sup \mathcal{E}]$ such that the level set

$$\mathcal{S}(c) := \{(\alpha_1, \alpha_2) \in L : \mathcal{E}(\alpha_1, \alpha_2) = c\} \subset \operatorname{int} L$$

(see Figure 2). Proceeding as in the proof of Proposition 4, one can find a smooth function $F\colon \mathbb{R}^2 \to \mathbb{R}$ and a number $\beta \in \mathbb{R}$ such that $F^{-1}\beta = \mathcal{S}(c) \subset \operatorname{int} L$. It follows from Theorem 6 that

$$\begin{aligned} h_{\mathrm{top}}(\sigma|C_\beta^F) &= \max_{(\alpha_1,\alpha_2)\in F^{-1}\beta} h_{\mathrm{top}}(\sigma|C_{(\alpha_1,\alpha_2)}) \\ &= \max_{(\alpha_1,\alpha_2)\in \mathcal{S}(c)} \mathcal{E}(\alpha_1,\alpha_2) = c = \mathcal{E}(\alpha_1^*, \alpha_2^*) \end{aligned}$$

for every $(\alpha_1^*, \alpha_2^*) \in \mathcal{S}(c)$. In this case, each point of $\mathcal{S}(c)$ maximizes the entropy spectrum, and so is associated to a distinct nonlinear full measure for β. Since $\mathcal{S}(c)$ is the closed curve obtained from $\mathcal{E}(x,y) = c$, we have an uncountable number of nonlinear full measures for the level set C_β^F.

5. Regularity of the nonlinear dimension spectrum

In this section we consider the problem of regularity of the nonlinear dimension spectrum. More precisely, we study the case when F is a (nonlinear) function defined on an open set containing the domain $L \subset \mathbb{R}^d$ and taking values in $\mathbb{R}^d$. For convenience of notation, we shall write $\mathcal{D}_u^F(\beta) = \dim_u C_\beta^F$, and in particular, when $u = 1$, we shall write $\mathcal{E}^F(\beta) = h_{\mathrm{top}}(T|C_\beta^F)$. We shall also denote the linear spectra by

$$\mathcal{D}_u(\alpha) = \dim_u C_\alpha \quad \text{and} \quad \mathcal{E}(\alpha) = h_{\mathrm{top}}(T|C_\alpha).$$

Based on Theorem 3, the domain of the nonlinear spectrum is the set

$$\{\beta \in \mathbb{R}^d : F^{-1}\beta \cap L \subset \operatorname{int} L\}.$$

Theorem 9. *Under the hypotheses of Theorem 3, let $F\colon U \supset L \to \mathbb{R}^d$ be a C^k function on an open set U with $k \geq 1$ or $k = \omega$ (real-analytic). Then for each β satisfying $F^{-1}\beta \cap L \subset \operatorname{int} L$ and such that the Jacobian matrix of F restricted to $F^{-1}\beta$ is invertible, there exist a neighborhood V of β and a C^k function $\widetilde{F}\colon V \to \mathbb{R}^d$ such that*

$$\mathcal{D}_u^F = \mathcal{D}_u \circ \widetilde{F}.$$

In particular, if the linear dimension spectrum $\mathcal{D}_u$ is C^k (respectively continuous) on $\operatorname{int} L$, the nonlinear dimension spectrum $\mathcal{D}_u^F$ is C^k (respectively continuous) on V.

Proof. Take $\beta \in \mathbb{R}^d$ such that $F^{-1}\beta \cap L \subset \operatorname{int} L$ and consider the function H given by $H(\beta, x) = F(x) - \beta$. For any $\alpha^* \in F^{-1}\beta$, we have $H(\beta, \alpha^*) = 0$ and, by hypothesis, the matrix $\partial_2 H(\beta, \alpha^*)$ is invertible. By the implicit function theorem, there exist an open set $V \subset \mathbb{R}^d$ containing β and a C^k function $\widetilde{F}\colon V \to \mathbb{R}^d$ such that $\widetilde{F}(\beta) = \alpha^*$ and $H(z, \widetilde{F}(z)) = 0$ for all $z \in V$. Together with Theorem 3 this implies that

$$\mathcal{D}_u^F(\beta) = \max_{\alpha \in F^{-1}\beta} \mathcal{D}_u(\alpha) = \mathcal{D}_u(\widetilde{F}(\beta)) \quad \text{with } F(\widetilde{F}(z)) = z \text{ for all } z \in V.$$

If necessary, one can consider neighborhoods $\widetilde{V} \subset V$ and $W \subset \widetilde{V}$ such that $\widetilde{F}(W) \subset F^{-1}\widetilde{V} \subset \operatorname{int} L$. Then the regularity of F and of the linear spectrum on $\operatorname{int} L$ can be used to obtain the regularity of the nonlinear spectrum on W. ☐

Remark 4. Theorem 9 includes of course the case when F is invertible. Moreover, when T is a subshift of finite type, an Axiom A $C^{1+\gamma}$ diffeomorphism or a $C^{1+\gamma}$ expanding map, assumed to be topologically mixing, and the functions in the collection $\Phi = \{\phi_1, \ldots, \phi_d\}$ are Hölder continuous, it follows from Theorem 13 in [7] that the dimension spectrum $\mathcal{D}_u$ is analytic on $\operatorname{int} L$. In these situations, with F

as in Theorem 9, the nonlinear dimension spectrum is also analytic on its domain (see Example 4). On the other hand, when the hypothesis on the Jacobian fails, even discontinuities may appear (see Example 5).

6. Finer structure of the level sets

In this section we explore some connections between the multifractal spectra of level sets with divergent points and the nonlinear thermodynamic formalism introduced in [12].

We say that a function $g\colon D \subset \mathbb{R}^d \to \mathbb{R} \cup \{-\infty\}$ is *concave of C^r Legendre type* for some $1 \le r \le \omega$ (where ω stands for real-analytic) if:

- g is upper semicontinuous;
- D has nonempty interior and g is strictly concave and C^r on $\operatorname{int} D$; moreover, when $r \ge 2$, the Hessian of g is negative-definite everywhere;
- for all sequences $(x_n)_{n\in\mathbb{N}}$ with $x_n \in \operatorname{int} D$ converging to a point in ∂D, we have $\lim_{n\to\infty} |\nabla g(x_n)| = +\infty$.

Let $T\colon X \to X$ be a continuous map of a compact metric space with finite topological entropy and let $\Phi = \{\phi_1, \ldots, \phi_d\}$ be a collection of continuous functions $\phi_j\colon X \to \mathbb{R}$. We consider the *entropy function* $h\colon \mathbb{R}^d \to \mathbb{R} \cup \{-\infty\}$ given by

$$h(\alpha) = \sup_{\mu\in\mathcal{M}(T)} \left\{ h_\mu(T) : \int_X \Phi \, d\mu = \alpha \right\}$$

and the *u-dimension function* $d_u\colon \mathbb{R}^d \to \mathbb{R} \cup \{-\infty\}$ given by

$$d_u(\alpha) = \sup_{\mu\in\mathcal{M}(T)} \left\{ \frac{h_\mu(T)}{\int_X u \, d\mu} : \int_X \Phi \, d\mu = \alpha \right\}.$$

With the usual convention that $\sup \emptyset = -\infty$ one can verify that, respectively,

$$L = \{\alpha \in \mathbb{R}^d : h(\alpha) \ne -\infty\} \quad \text{and} \quad L = \{\alpha \in \mathbb{R}^d : d_u(\alpha) \ne -\infty\}.$$

In particular, under the hypotheses of Theorem 2, we have

$$h(\alpha) = h_{\text{top}}(T|C_\alpha) \quad \text{and} \quad d_u(\alpha) = \dim_u C_\alpha \quad \text{for all } \alpha \in \operatorname{int} L. \tag{13}$$

Following [12], we say that the pair (T, Φ) is C^r *Legendre* with $1 \le r \le \omega$ if:

- the set $L \subset \mathbb{R}^d$ has nonempty interior;
- $h_{\text{top}}(T) < \infty$;
- the function h is concave of C^r Legendre type.

In what follows, we shall write (T, u, Φ) to refer to $\{u, \Phi\}$ as the collection of functions $\{u, \phi_1, \ldots, \phi_d\}$.

Denote by $C_T(X) \subset C^0(X)$ the family of continuous functions $\varphi\colon X \to \mathbb{R}$ for which given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|S_n\varphi(x) - S_n\varphi(y)| < \varepsilon \quad \text{whenever } d_n(x, y) < \delta,$$

where

$$d_n(x, y) = \max\bigl\{ d(T^k(x), T^k(y)) : k = 0, \ldots, n-1 \bigr\}.$$

Let T be a topologically mixing subshift of finite type, an Axiom A $C^{1+\gamma}$ diffeomorphism or a $C^{1+\gamma}$ expanding map. When $\Phi = \{\phi_1, \ldots, \phi_d\}$ is such that $\phi_j \in C_T(X)$ for all $1 \le j \le d$ and the cohomology classes of the functions $1, \phi_1, \ldots, \phi_d$ are linearly independent, then (T, Φ) is a C^1 Legendre pair. Furthermore, when each ϕ_j is Hölder continuous the pair (T, Φ) is C^ω Legendre (see [12, Proposition 3.10], [7, Theorem 12] and [7, Theorem 13]).

Proposition 10. *Let (T,u,Φ) be C^1 Legendre and let $F\colon U\to\mathbb{R}$ be a continuous function, where U is an open set containing L. Then*

$$\sup_{\alpha\in L}\bigl\{d_u(\alpha)+F(\alpha)\bigr\}=\max_{\alpha\in\operatorname{int}L(\Phi)}\bigl\{d_u(\alpha)+F(\alpha)\bigr\}>\max_{\alpha\in\partial L}\bigl\{d_u(\alpha)+F(\alpha)\bigr\}.$$

In particular, the maximum of d_u is only attained on $\operatorname{int}L$.

Proof. Following [12], we define the *generalized nonlinear energy* by

$$\Pi^G(\mu,u,\Phi)=G\biggl(h_\mu(T),\int_X u\,d\mu,\int_X\Phi\,d\mu\biggr),$$

where $G\colon V\to\mathbb{R}$ is any continuous function on an open set $V\subset\mathbb{R}\times\mathbb{R}^{d+1}$ satisfying the conditions

$$\partial_{x_0}G(x_0,x_1,\dots,x_{d+1})>0,\quad\biggl\{\biggl(h_\mu(T),\int_X u\,d\mu,\int_X\Phi\,d\mu\biggr):\mu\in\mathcal{M}(T)\biggr\}\subset V. \tag{14}$$

Let also

$$L_u=\biggl\{\biggl(\int_X u\,d\mu,\int_X\Phi\,d\mu\biggr):\mu\in\mathcal{M}(T)\biggr\}\subset\mathbb{R}\times\mathbb{R}^d.$$

When a measure $\eta\in\mathcal{M}(T)$ maximizes the map $\mu\mapsto\Pi^G(\mu,u,\Phi)$ and (T,u,Φ) is C^1 Legendre, it follows from Claim 3 in [12] that

$$\Pi^G(\eta,u,\Phi)=\sup_{(z_1,\dots,z_{d+1})\in\operatorname{int}L_u}G(h(z_1,\dots,z_{d+1}),z_1,\dots,z_{d+1}) \tag{15}$$

and that the maximum is never attained on ∂L_u. Now we consider the function $\widetilde{G}\colon\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^d\to\mathbb{R}$ given by

$$\widetilde{G}(x_0,x_1,x_2,\dots,x_{d+1})=\frac{x_0}{x_1}+F(x_2,\dots,x_{d+1}),$$

where $F\colon U\subset\mathbb{R}^d\to\mathbb{R}$ is any continuous function on some open set $U\supset L$. Since $\partial_{x_0}\widetilde{G}=1/x_1>0$, the function $\widetilde{G}$ satisfies (14). Hence, by (15) we obtain

$$\begin{aligned}
&\sup_{\mu\in\mathcal{M}(T)}\biggl\{\frac{h_\mu(T)}{\int_X u\,d\mu}+F\biggl(\int_X\Phi\,d\mu\biggr)\biggr\}\\
&=\sup_{(z_1,\dots,z_{d+1})\in\operatorname{int}L_u}\biggl\{\frac{h(z_1,\dots,z_{d+1})}{z_1}+F(z_2,\dots,z_{d+1})\biggr\}\\
&=\frac{h(z_1^*,z_2^*,\dots,z_{d+1}^*)}{z_1^*}+F(z_2^*,\dots,z_{d+1}^*)
\end{aligned}$$

for some $(z_1^*,z_2^*,\dots,z_{d+1}^*)\in\operatorname{int}L_u$. In other words, there exists $\nu\in\mathcal{M}(T)$ with $\int_X u\,d\nu=z_1^*$ and $\int_X\Phi\,d\nu=(z_2^*,\dots,z_{d+1}^*)$ such that

$$\sup_{\mu\in\mathcal{M}(T)}\biggl\{\frac{h_\mu(T)}{\int_X u\,d\mu}+F\biggl(\int_X\Phi\,d\mu\biggr)\biggr\}=\frac{h_\nu(T)}{\int_X u\,d\nu}+F\biggl(\int_X\Phi\,d\nu\biggr).$$

Therefore,

$$\sup_{\alpha\in L}\bigl\{d_u(\alpha)+F(\alpha)\bigr\}=\frac{h_\nu(T)}{\int_X u\,d\nu}+F\biggl(\int_X\Phi\,d\nu\biggr)=\max_{\alpha\in\operatorname{int}L}\bigl\{d_u(\alpha)+F(\alpha)\bigr\}$$

with $\int_X\Phi\,d\nu\in\operatorname{int}L$, which completes the proof. □

In a similar direction, we also have the following result. Let $\mathcal{H}_\theta$ be the set of real-valued Hölder continuous functions with exponent θ.

Theorem 11 ([7, Theorem 14]). *Let T be a topologically mixing subshift of finite type, a $C^{1+\gamma}$ diffeomorphism with a hyperbolic set, or a $C^{1+\gamma}$ map with a repeller. Then there exists a residual set $\mathcal{O} \subset (\mathcal{H}_\theta)^d$ such that for each $\Phi \in \mathcal{O}$ we have*

$$d_u|\partial L \equiv 0 \quad \text{and} \quad L = \overline{\operatorname{int} L}. \tag{16}$$

In particular, either $d_u \equiv 0$ or d_u does not attain a maximum on ∂L.

Motivated by Theorem 11, we say that a pair (T, Φ) is *typical* if it satisfies (16).

The next result gives a relation between the dimension spectrum and the nonlinear topological pressure, and shows how the maximizing points for the linear dimension spectrum on L play an important role on the "size" of level sets with divergent points.

Theorem 12. *Let $T\colon X \to X$ be a continuous map of a compact metric space such that the metric entropy $\mu \mapsto h_\mu(T)$ is upper semicontinuous and $h_{\text{top}}(T) < \infty$. Suppose that $u\colon X \to \mathbb{R}$ is a strictly positive continuous function, $\Phi = \{\phi_1, \ldots, \phi_d\}$ is a collection of real-valued continuous functions on X and assume that there exists a dense subspace $D(X) \subset C^0(X)$ such that every $\xi \in D(X)$ has a unique equilibrium measure. Then the following properties hold:*

1. *If in addition (T, u, Φ) is C^1 Legendre or (T, Φ) is typical, then*

$$\max_{\alpha \in \operatorname{int} L} \dim_u C_\alpha = \dim_u X.$$

In particular, a compact set $A \subset \operatorname{int} L$ contains a maximizing point for the u-dimension function $d_u|L$ if and only if

$$\dim_u C_A = \dim_u X.$$

2. *If in addition (T, Φ) is C^1 Legendre with an abundance of ergodic measures, and the functions h and d_u are continuous on L, then*

$$\max_{\alpha \in \operatorname{int} L} \dim_u C_\alpha = P_{G_u}(\Phi) = \dim_u X, \tag{17}$$

where $G_u\colon \mathbb{R}^d \to \mathbb{R}$ is a continuous function such that $G_u(\alpha)|L = d_u(\alpha) - h(\alpha)$ and $P_{G_u}(\Phi)$ is the nonlinear topological pressure for (G_u, Φ). In particular, a compact set $A \subset \operatorname{int} L$ contains a maximizing point for the u-dimension function $d_u|L$ if and only if

$$\dim_u C_A = P_{G_u}(\Phi) = \dim_u X.$$

3. *All the maximizing points for the u-dimension function are in ∂L if and only if for each compact set $A \subset \operatorname{int} L$ we have*

$$\dim_u C_A < \max_{\alpha \in L} d_u(\alpha) = \dim_u X.$$

In particular, (T, u, Φ) is not C^1 Legendre.

Proof. The first item follows from combining Proposition 10, the definition of typical pairs and Theorem 2. Now we prove item 2. Consider the continuous function $\widetilde{G_u}(\alpha) = d_u(\alpha) - h(\alpha)$ for $\alpha \in L$. Since $L \subset \mathbb{R}^d$ is compact, one can find a continuous function $G_u\colon \mathbb{R}^d \to \mathbb{R}$ such that $G_u|L = \widetilde{G_u}|L$ (see for example Theorem 4.16 in [20]). Since (T, Φ) is C^1 Legendre, it follows from Theorem 3.16 in [12] that every maximizing point of $h + G_u$ belongs to $\operatorname{int} L$. By Theorem 2, $d_u(\alpha) = \dim_u C_\alpha$ for all $\alpha \in \operatorname{int} L$, and we obtain

$$\max_{\alpha \in L} d_u(\alpha) = \max_{\alpha \in \operatorname{int} L} d_u(\alpha) = \max_{\alpha \in \operatorname{int} L} \dim_u C_\alpha. \tag{18}$$

Now consider the set $\mathcal{M}_\alpha = \{\mu \in \mathcal{M}(T) : \int_X \Phi \, d\mu = \alpha\}$. Since (T,Φ) has an abundance of ergodic measures, we can apply the nonlinear variational principle to obtain

$$\begin{aligned} P_{G_u}(\Phi) &= \sup_{\eta\in\mathcal{M}(T)} \left\{ h_\eta(T) + G_u\left(\int_X \Phi \, d\eta\right)\right\} \\ &= \sup_{\alpha\in L}\sup_{\eta\in\mathcal{M}_\alpha} \left\{ h_\eta(T) + G_u\left(\int_X \Phi \, d\eta\right)\right\} \\ &= \sup_{\alpha\in L}\sup_{\eta\in\mathcal{M}_\alpha} \{h_\eta(T) + G_u(\alpha)\} \\ &= \sup_{\alpha\in L}\{h(\alpha) + G_u(\alpha)\} = \max_{\alpha\in L} d_u(\alpha). \end{aligned} \tag{19}$$

By the definitions of classical topological pressure and u-dimension, one can see that $P_X(-\dim_u X u) = 0$. Hence, it follows from (18) that

$$\dim_u X = \frac{h_\nu(T)}{\int_X u \, d\nu} \le \max_{\alpha\in L} d_u(\alpha) = \max_{\alpha\in \operatorname{int} L} \dim_u C_\alpha,$$

where $\nu \in \mathcal{M}(T)$ is an equilibrium measure for the function $-\dim_u X u$. By Theorem A3.1 in [30], we have $\dim_u C_\alpha \le \dim_u X$ for each $\alpha \in \operatorname{int} L$. Hence, it follows that $\dim_u X = \max_{\alpha\in\operatorname{int} L} \dim_u C_\alpha$. Together with (18) and (19), this establishes identity (17).

By Theorem 2, for each compact set $A \subset \operatorname{int} L$, we have $\dim_u C_A = \max_{\alpha\in A} C_\alpha$. In this case, it follows from (13) that

$$\dim_u C_A = \max_{\alpha\in A} \dim_u C_\alpha = \max_{\alpha\in A} d_u(\alpha).$$

Hence, if a set A contains a maximizing point α^* for the function $d_u|L$, then we get

$$\dim_u C_A = \dim_u C_{\alpha^*} = d_u(\alpha^*) = \dim_u X.$$

On the other hand, when a set A contains no maximizing point for d_u on L, then

$$\dim_u C_A < \max_{\alpha\in L} d_u(\alpha) = \dim_u X.$$

To prove item 3, note that if there exists $A \subset \operatorname{int} L$ with $\dim_u C_A = \dim_u X$, then there is a point $\alpha^* \in A$ such that $\dim_u C_{\alpha^*} = \dim_u X = \max_{\alpha\in L} d_u(\alpha)$. In other words, $\alpha^* \in \operatorname{int} L$ is a maximizing point for $d_u|L$. □

Remark 5. Note that the equivalence between level sets with distinct limit points and nonlinear levels sets described in Lemma 1 allows one to give the same type of information as in Theorem 12, although now with respect to nonlinear level sets (see Section 8).

The first item in Theorem 12 gives two different types of setups (that may intersect) where one can find proper level sets of full dimension, and this property only depends on the localization of the maximizing points for the u-dimension function (or the entropy function) on L. Therefore, the dimensions of the sets C_A do not depend necessarily on the shape or "size" of the sets $A \subset L$ (see Figure 3). Analogously, the second item in Theorem 12 gives a relation between the u-dimension and the nonlinear topological pressure, and indicates that for C^1 Legendre pairs with an abundance of ergodic measures, some level sets with divergent points can have full dimension (or full entropy). Finally, the third item in Theorem 12 describes a similar dependence on the localization of the maxima of the u-dimension function, for more general systems (those satisfying the hypotheses of Theorem 2).

We already saw that many classical hyperbolic and expanding dynamical systems satisfy the C^1 Legendre property. Now we consider some examples of pairs with an abundance of ergodic measures. We first recall an important related notion. A

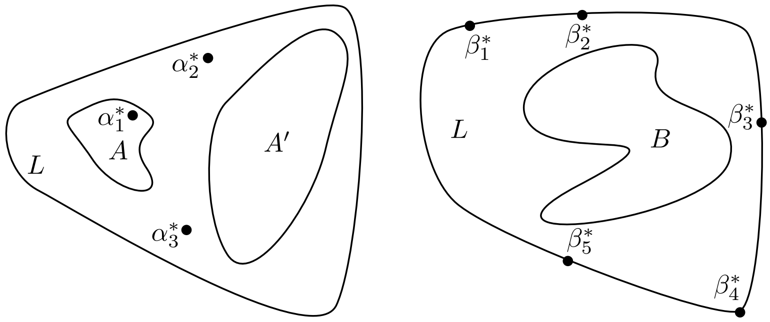

FIGURE 3. On the left: all the maximizing points $\{\alpha_1^*, \alpha_2^*, \alpha_3^*\}$ for the u-dimension function are in $\operatorname{int} L$. In this case, $\dim_u X = \dim_u C_A > \dim_u C_{A'}$. On the right: all the maximizing points $\{\beta_1^*, \beta_2^*, \beta_3^*, \beta_4^*, \beta_5^*\}$ for the u-dimension function are in ∂L. In this case, $\dim_u X > \dim_u C_B$.

map T is said to have *entropy density of ergodic measures* if for every $\mu \in \mathcal{M}(T)$ there exist ergodic measures $\nu_n \in \mathcal{M}(T)$ for $n \in \mathbb{N}$ such that $\nu_n \to \mu$ in the weak* topology and $h_{\nu_n}(T) \to h_\mu(T)$ when $n \to \infty$. If T has entropy density of ergodic measures, then the pair (T, Φ) has an abundance of ergodic measures for any family of continuous functions Φ. It is known that mixing subshifts of finite type and mixing locally maximal hyperbolic sets have entropy density of ergodic measures (see [18, Theorem B] and [32, Theorem 2.1]). In addition, any continuous map T of a compact metric space with the weak specification property such that the entropy map $\mu \mapsto h_\mu(T)$ is upper semicontinuous also has entropy density of ergodic measures (see [15]). As a consequence, some examples of pairs (T, Φ) with an abundance of ergodic measures include expansive maps with specification or weak specification, topologically mixing diffeomorphisms with locally maximal hyperbolic sets, subshifts of finite type, sofic shifts and β-shifts, together with any family of continuous functions Φ.

Observe that in item 2 of Theorem 12 we required the entropy and u-dimension functions to be continuous on L, although in general these are only upper semicontinuous. This issue was considered in the literature in the context of *rotation theory*, where $h|L$ is also called the *localized entropy function*. It was showed in [21] that for a compact convex set $K \subset \mathbb{R}^d$, the property that every concave upper semicontinuous function on K is continuous is equivalent to K being a polyhedron. We note that L is a polyhedron for subshifts of finite type with locally constant potentials, and in fact with certain nonlocally constant potentials (see [23, 26, 41]). This guarantees the continuity of h on L. Furthermore, in the same setups, if d_u is concave on L, then it is also continuous. In the one-dimensional case, it is easy to see that every concave upper semicontinuous function on $L \subset \mathbb{R}$ is continuous on L. On the other hand, as showed in [40], this is not always the case in higher dimensions, where h may be discontinuous at the boundary of L even for the full shift and Lipschitz continuous functions.

7. Nontypical points and full measures

Using the existence of nonlinear full measures obtained in Section 3.1, here we study the u-dimension of *nonlinear irregular sets* while extending several results in [8]. The main element of our approach is that nonlinear full measures can be approximated by or are in fact *distinguishing measures* (as introduced in that paper).

Let $T\colon X \to X$ be a continuous map of a compact metric space, $\Psi = \{\psi_1, \ldots, \psi_d\}$ a collection of real-valued continuous functions, and $F\colon U \subset \mathbb{R}^d \to \mathbb{R}^p$ a continuous function, where U is an open set containing L. We define the *nonlinear irregular set* (with respect to F) by

$$\mathcal{I}_F = \left\{x \in X : \lim_{n\to\infty} F\left(\frac{1}{n}S_n\psi_1(x), \ldots, \frac{1}{n}S_n\psi_d(x)\right) \text{ does not exist}\right\}.$$

The set X can be decomposed in the form

$$X = \mathcal{I}_F \cup \bigcup_{\beta\in\mathbb{R}^p} C_\beta^F,$$

a nonlinear counterpart of the *multifractal decomposition* of X (see [30] and references within). Considering the (individual linear) irregular sets

$$\mathcal{B}(\psi_j) = \left\{x \in X : \lim_{n\to\infty} \frac{1}{n}S_n\psi_j(x) \text{ does not exist}\right\},$$

one can check that

$$\mathcal{I}_F \subset \mathcal{B}(\psi_1) \cup \cdots \cup \mathcal{B}(\psi_d).$$

Moreover, we note that in general $\mathcal{B}(\psi_1) \cap \cdots \cap \mathcal{B}(\psi_d)$ may not be contained in $\mathcal{I}_F$. This means that, a priori, the nonlinear irregular sets may be smaller than the linear irregular sets. A natural question is thus how large is $\mathcal{I}_F$ from the point of view of dimension and, more specifically, under what hypotheses one can show that $\dim_u \mathcal{I}_F = \dim_u X$.

The following notion was introduced in [8].

Definition 1. Let $G^i = \{g_n^i\colon X \to \mathbb{R}\}_{n\in\mathbb{N}}$ for $i = 1, \ldots, m$ be sequences of continuous functions. A collection of measures $\{\mu_1, \ldots, \mu_k\}$ on X is called *distinguishing* for $G^1, \ldots, G^m$ if for every $1 \le i \le m$ there exist distinct integers $j_1 = j_1(i)$ and $j_2 = j_2(i)$ in $[1, k]$ and numbers $a_{j_1}^i \ne a_{j_2}^i$ such that

$$\lim_{n\to\infty} g_n^i(x) = a_{j_1}^i \quad \text{for } \mu_{j_1}\text{-almost every } x \in X,$$
$$\lim_{n\to\infty} g_n^i(x) = a_{j_2}^i \quad \text{for } \mu_{j_2}\text{-almost every } x \in X.$$

We also need the following result.

Theorem 13 ([8, Theorem 7.2])**.** *Let $\sigma\colon X \to X$ be a one-sided or two-sided subshift satisfying the specification property and let $\{\mu_1, \ldots, \mu_k\}$ be a collection of ergodic σ-invariant distinguishing measures for $G^1, \ldots, G^m$. If $u\colon X \to \mathbb{R}$ is a strictly positive Hölder continuous function, then*

$$\dim_u \mathcal{B}(G^1, \ldots, G^m) \ge \min\{\dim_u \mu_1, \ldots, \dim_u \mu_k\},$$

where

$$\mathcal{B}(G^1, \ldots, G^m) = \left\{x \in X : \lim_{n\to\infty} g_n^i(x) \text{ does not exist for } i = 1, \ldots, m\right\}.$$

Based on Theorem 12, we introduce a property that is relevant for our purposes. Let (T, u, Φ) be C^1 Legendre. Then the first item in Theorem 12 guarantees the existence of $\alpha^* \in \operatorname{int} L$ such that $\dim_u C_{\alpha^*} = \dim_u X$. We say that a function $F\colon U \supset L \to \mathbb{R}^p$ satisfies the *void property with respect to* (T, u, Φ) if the parameter $\beta \in \mathbb{R}^p$ with $\alpha^* \in F^{-1}\beta$ is such that $\operatorname{int} F^{-1}\beta = \emptyset$ (with respect to the norm of $\mathbb{R}^d$). For example, every C^1 function without critical points satisfies the void property with respect to any C^1 Legendre pair since then any set $F^{-1}\beta \subset \mathbb{R}^d$ is a manifold of dimension $d - 1$.

Now we state the main result of this section.

Theorem 14. *Let $\sigma\colon X \to X$ be a one-sided or two-sided topologically mixing subshift of finite type, $\Psi = \{\psi_1, \dots, \psi_d\}$ a collection of real-valued Hölder continuous functions on X and $u\colon X \to \mathbb{R}^+$ a Hölder continuous function such that the cohomology classes of $1, u, \psi_1, \dots, \psi_d$ are linearly independent. Suppose that $F = (F_1, \dots, F_p)\colon U \supset L \to \mathbb{R}^p$ is such that each $F_k\colon U \to \mathbb{R}$ is a continuous function satisfying the void property with respect to (T, u, Ψ), where $U \subset \mathbb{R}^d$ is an open set. Then*

$$\dim_u \mathcal{I}_F = \dim_u X.$$

Proof. Write

$$F(x_1, \dots, x_d) = \big(F_1(x_1, \dots, x_d), \dots, F_p(x_1, \dots, x_d)\big) \quad \text{for } (x_1, \dots, x_d) \in U,$$

and for each $k = 1, \dots, p$ define the nonlinear irregular sets

$$\mathcal{I}_{F_k} = \left\{ x \in X : \limsup_{n\to\infty} F_k\left(\frac{1}{n} S_n\Psi(x)\right) > \liminf_{n\to\infty} F_k\left(\frac{1}{n} S_n\Psi(x)\right)\right\}.$$

Then clearly

$$\mathcal{I}_{F_k} \subset \mathcal{I}_F \subset \bigcup_{j=1}^{p} \mathcal{I}_{F_j} \quad \text{for every } k \in \{1, \dots, p\} \tag{20}$$

and whenever $\mathcal{I}_F \neq \emptyset$, there exists $\ell \in \{1, \dots, p\}$ such that $\mathcal{I}_{F_\ell} \neq \emptyset$.

Since every Hölder continuous function has a unique equilibrium measure with respect to any topologically mixing subshift of finite type and (σ, u, Ψ) is a C^1 Legendre pair, one can apply Theorems 6 and 12. We consider the sequence of real-valued functions $G^\ell = (G_n^\ell)_{n\in\mathbb{N}}$ given by $G_n^\ell(x) = F_\ell\big(\frac{1}{n} S_n\Psi(x)\big)$ for every $x \in X$ and $n \in \mathbb{N}$. By the first item in Theorem 12 and Theorem 6, there exist $\alpha_1^* \in \operatorname{int} L$ and a σ-invariant ergodic measure $\nu_{\alpha_1^*}$ such that

$$\int_X \Psi \, d\nu_{\alpha_1^*} = \alpha_1^* \quad \text{and} \quad \dim_u \nu_{\alpha_1^*} = \dim_u C_{\alpha_1^*} = \dim_u X. \tag{21}$$

Take $\beta_1 = F(\alpha_1^*)$ and $\varepsilon > 0$. Since the linear spectrum $\alpha \mapsto \dim_u C_\alpha$ is continuous on $\operatorname{int} L$ and, by assumption, F_ℓ satisfies the void property with respect to (T, u, Ψ), there exist $\delta > 0$ and $\alpha_2^* \in B(\alpha_1^*, \delta)/F_\ell^{-1}\beta_1 \subset \operatorname{int} L$ such that $\dim_u C_{\alpha_2^*} = \dim_u X - \varepsilon/2$. By Theorem 6, there exists an ergodic σ-invariant measure $\nu_{\alpha_2^*}$ such that

$$\int_X \Psi \, d\nu_{\alpha_2^*} = \alpha_2^* \quad \text{and} \quad \dim_u \nu_{\alpha_2^*} = \dim_u C_{\alpha_2^*} \geq \dim_u X - \varepsilon. \tag{22}$$

Now let $\beta_2 = F_\ell(\alpha_2^*)$. By Birkhoff's Ergodic Theorem and the continuity of F_ℓ, we have

$$\lim_{n\to\infty} G_n^\ell(x) = F_\ell\left(\int_X \Psi \, d\nu_{\alpha_1^*}\right) = \beta_1 \quad \text{for } \nu_{\alpha_1^*}\text{-almost every } x \in X,$$

$$\lim_{n\to\infty} G_n^\ell(x) = F_\ell\left(\int_X \Psi \, d\nu_{\alpha_2^*}\right) = \beta_2 \quad \text{for } \nu_{\alpha_2^*}\text{-almost every } x \in X.$$

Since $\alpha_2^* \notin F_\ell^{-1}\beta_1$, we have $\beta_2 \neq \beta_1$, which readily implies that $\{\nu_{\alpha_1^*}, \nu_{\alpha_2^*}\}$ is a collection of distinguishing measures for the sequence G^ℓ. Hence, it follows directly from Theorem 13 together with (20), (21) and (22) that

$$\dim_u \mathcal{I}_F \geq \dim_u \mathcal{I}_{F_\ell} \geq \min\{\dim_u \nu_{\alpha_1^*}, \dim_u \nu_{\alpha_2^*}\} \geq \dim_u X - \varepsilon.$$

The arbitrariness of ε guarantees that $\dim_u \mathcal{I}_F = \dim_u X$. □

Remark 6. The void property allows one to avoid some pathological cases, where there might exist $\varepsilon > 0$ and $\beta \in \mathbb{R}^p$ such that if $\dim_u C_\alpha \geq \dim_u X - \varepsilon$, then $\alpha \in F^{-1}\beta$. In these cases, the continuity of the linear spectrum $\alpha \mapsto \dim_u C_\alpha$ on $\operatorname{int} L$ forces that $\operatorname{int} F^{-1}\beta \neq \emptyset$.

Remark 7. It follows from the proof of Theorem 14 that if the set $F(\operatorname{int} L) \subset \mathbb{R}^p$ has more than one element, then the existence of distinguishing measures implies that $\mathcal{I}_F \neq \emptyset$. Moreover, when $F_\ell^{-1}\beta_1, F_\ell^{-1}\beta_2 \subset \operatorname{int} L$, it follows from Theorem 6 that $\dim_u C_{\beta_j}^{F_\ell} = \dim_u \nu_{\alpha_j^*}$ for $j = 1, 2$, in which case the collection of distinguishing measures $\{\nu_{\alpha_1^*}, \nu_{\alpha_2^*}\}$ for G^ℓ is composed of nonlinear full measures.

Remark 8. Some related nonlinear multifractal problems were discussed by Huang, Tian and Wang in [22], although using different methods. As noted in Remarks 4.3 and 6.2 in [22], their approach cannot decide, for instance, if the irregular set $\mathcal{I}_F$ has full topological entropy (with respect to the full shift), where $\Phi = \{\phi_1, \phi_2\}$ is a collection of real-valued Hölder continuous functions on $\Sigma^{\mathbb{N}}$ and $F(x, y) = xy$ (this nonlinear function also plays an important role in Section 9). However, since F satisfies the void property, we can apply Theorem 14 (with $u \equiv 1$) to show in particular that $h_{\text{top}}(\sigma|\mathcal{I}_F) = h_{\text{top}}(\sigma)$.

We conclude this section by observing that using Markov partitions Theorem 14 can be properly adapted to give corresponding results for uniformly expanding and hyperbolic maps (see [8]). Moreover, in the conformal case, we also have versions of Theorem 14 for the Hausdorff dimension (see Theorem 16).

8. Some applications to frequencies of digits

In this section we use the nonlinear conditional variational principles to study some classes of entropy and dimension spectra. These are related to level sets defined in terms of frequencies with respect to symbolic dynamics and the representation of numbers in different bases.

8.1. **Frequency of symbols.** We give some explicit examples in the context of symbolic dynamics. We start with the case of a one-dimensional concave nonlinear dimension spectrum for the full shift, when the nonlinear function F is invertible.

Given a dynamical system $T\colon X \to X$, we define the *frequency of a point x in a set $E \subset X$* by

$$\tau(x, E) = \lim_{n\to\infty} \frac{1}{n}\#\{0 \le j < n : T^j(x) \in E\} = \lim_{n\to\infty} \sum_{k=0}^{n-1} 1_E(T^k(x)).$$

Example 4. Consider $\Sigma = \{0, 1\}$, the shift map $\sigma\colon \Sigma^{\mathbb{N}} \to \Sigma^{\mathbb{N}}$, the potential $\varphi = 1_{C_1}$, the function $u(\omega_0\omega_1\dots) = u_{\omega_0}$ with $u_0 = 2$ and $u_1 = 3$, and the nonlinear function $F\colon \mathbb{R} \to \mathbb{R}$ given by $F(x) = \cos x$. In this case, $L = [0, 1]$ and

$$C_\beta^F = \{\omega \in \Sigma^{\mathbb{N}} : \cos\tau(\omega, C_1) = \beta\}.$$

By Theorem 6 we obtain

$$\begin{aligned}
\mathcal{D}_u^F(\beta) = \dim_u C_\beta^F &= \max_{\mu\in\mathcal{M}(\sigma)} \left\{\frac{h_\mu(\sigma)}{\int_{\Sigma^{\mathbb{N}}} u\,d\mu} : \cos\mu(C_1) = \beta\right\} \\
&= \max_{\mu\in\mathcal{M}(\sigma)} \left\{\frac{h_\mu(\sigma)}{\mu(C_1) + 2} : \cos\mu(C_1) = \beta\right\} \\
&= \frac{1}{\arccos\beta + 2} \max_{\mu\in\mathcal{M}(\sigma)} \left\{h_\mu(\sigma) : \cos\mu(C_1) = \beta\right\} \\
&= \frac{-\arccos\beta\log(\arccos\beta) - (1 - \arccos\beta)\log(1 - \arccos\beta)}{\arccos\beta + 2}
\end{aligned}$$

for every $\beta \in [0, 1]$ satisfying $\arccos\beta \in (0, 1)$. Notice that $\beta \mapsto \mathcal{D}_u^F(\beta)$ is analytic and concave on its domain $(\cos 1, 1)$. Moreover, when $\beta^* \approx 0.909$, we have

$$0.281 \approx \dim_u \Sigma^{\mathbb{N}} = \mathcal{D}_u^F(\beta^*) = \dim_u C_{\beta^*}^F = \dim_u \eta_{\beta^*},$$

where η_{β^*} is the ergodic nonlinear full measure for β^*. In this case, the level set $C^F_{\beta^*}$ is the only one carrying full dimension. One can also verify that

$$F^{-1}\beta^* = \{\arccos\beta^*\} = \{\alpha^* \approx 0.430\} \subset \operatorname{int} L$$

with

$$\mathcal{D}_u(\alpha^*) = \dim_u \Sigma^{\mathbb{N}} = \mathcal{D}^F_u(\beta^*).$$

Now we describe a discontinuous nonlinear dimension spectrum.

Example 5. Consider $\Sigma = \{0, 1\}$, the shift map $\sigma\colon \Sigma^{\mathbb{N}} \to \Sigma^{\mathbb{N}}$, the constant function $u = 1/17$ and the potential $\phi = 15(1_{C_1})$. Let also $Q\colon \mathbb{R} \to \mathbb{R}$ be the cubic polynomial given by

$$Q(x) = \frac{1}{7}(x^3 - 18x^2 + 89x - 132) + 3.$$

Proceeding as before, one can see that $L = [0, 15]$ and that the linear dimension spectrum is given by

$$\mathcal{D}_u(\alpha) = -\frac{17\alpha}{15}\log\left(\frac{\alpha}{15}\right) - \left(17 - \frac{17\alpha}{15}\right)\log\left(1 - \frac{\alpha}{15}\right).$$

We observe that Q is not invertible on L. Moreover, it has the critical points $6 \pm \sqrt{19/3}$ on L (see Figure 4). Let $\alpha^* = 6 + \sqrt{19/3}$ and take $\beta_0 = Q(\alpha^*)$. Observe that $Q^{-1}\beta_0 = \{\alpha^*, \alpha'\}$, where $\alpha' \in (0, \alpha^*)$. It follows from Theorem 6 that

$$\mathcal{D}^Q_u(\beta_0) = \max_{\alpha \in Q^{-1}\beta_0} \mathcal{D}_u(\alpha) = \mathcal{D}_u(\alpha^*).$$

On the other hand, for each $\delta > 0$ we have

$$\mathcal{D}_u(\beta_0) - \mathcal{D}(\beta_0 - \delta) \geq \mathcal{D}_u(\alpha^*) - \mathcal{D}_u(\alpha') > 0.$$

Therefore, and as shown in Figure 4, $Q^{-1}\beta_0 \subset \operatorname{int} L$ and there is a jump from $\mathcal{D}^Q_u(\beta_0 - \delta)$ to $\mathcal{D}^Q_u(\beta_0)$. In other words, the nonlinear dimension spectrum is discontinuous at β_0. Incidentally, the other critical point of Q does not lead to a discontinuity for the nonlinear spectrum.

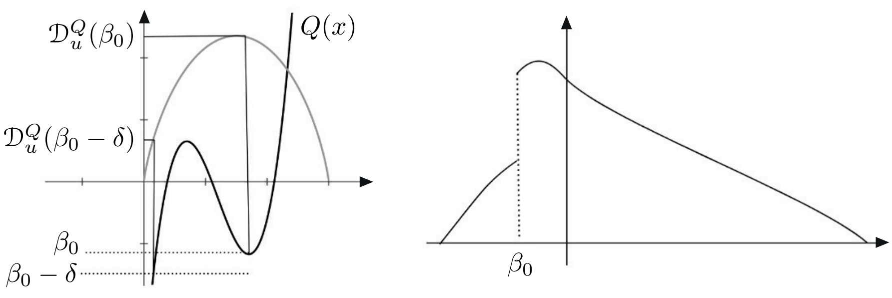

FIGURE 4. The graph of the nonlinear dimension spectrum $\mathcal{D}^Q_u$ with a discontinuity at β_0.

Note that in the former examples we have C^1 Legendre pairs, and so by Theorem 12 there exist nonlinear level sets of full dimension.

8.2. **Frequency of digits.** Now let us consider the uniformly expanding map $T_k\colon [0,1] \to [0,1]$ given by $T_k x = kx \bmod 1$ for some positive integer $k > 1$. We restrict our attention to 1-*locally constant functions* $\phi_j\colon [0,1] \to \mathbb{R}$, that is, functions that are constant on intervals of the form $[\ell/k, (\ell+1)/k)$. For convenience, we write $\phi_{j\ell} = \phi_j([\ell/k, (\ell+1)/k))$ and we consider the set

$$S_k = \big\{(\gamma_0, \dots, \gamma_{k-1}) \in [0,1]^k : \gamma_0 + \dots + \gamma_{k-1} = 1\big\}.$$

For a function $\psi\colon [0,1] \to \mathbb{R}$ given by $\psi(0.x_1x_2\dots) = a_{x_1}$ (with $0.x_1x_2\cdots$ written in base k) for some constants $a_j \in \mathbb{R}$ for $j \in \{0, ..., k\}$, we have

$$P(\psi) = \log \sum_{j=0}^{k-1} \exp a_j, \tag{23}$$

where $P(\psi)$ is the classical topological pressure of ψ with respect to T_k.

Using Theorem 6, we give a complete multifractal analysis for the Hausdorff dimension of the level sets involving nonlinear relations of frequencies of digits.

Theorem 15. *For the expanding map T_k, consider a collection $\Psi = \{\psi_1, \dots, \psi_d\}$ of 1-locally constant functions $\psi_j\colon [0,1] \to \mathbb{R}$ and let $F\colon U \to \mathbb{R}^p$ be a piecewise continuous function with finitely many discontinuities and at most at the positive integer powers of $1/k$, where U is an open set containing $L \subset \mathbb{R}^d$. Then the following properties hold:*

1. *if $C^F_\beta \neq \emptyset$, then*

$$\begin{aligned}\dim_H C^F_\beta &= \frac{1}{\log k} \sup_{\mu \in \mathcal{M}(T)} \left\{ h_\mu(T_k) : F\left(\int_0^1 \Psi\, d\mu\right) = \beta \right\} \\ &= \frac{1}{\log k} \sup_{\alpha \in F^{-1}\beta} \sup_{(\gamma_0,\dots,\gamma_{k-1}) \in \Gamma_k(\alpha)} \left\{ -\sum_{\ell=0}^{k-1} \gamma_\ell \log \gamma_\ell \right\},\end{aligned}$$

   where $\alpha = (\alpha_1, \dots, \alpha_d)$ and

$$\Gamma_k(\alpha) = \left\{ (\gamma_0, \dots, \gamma_{k-1}) \in S_k : \sum_{\ell=0}^{k-1} \gamma_\ell \psi_{j\ell} = \alpha_j \text{ for } j = 1, \dots, d \right\};$$

2. *if in addition F is continuous and $\beta \in \mathbb{R}^p$ satisfies $F^{-1}\beta \cap L \subset \operatorname{int} L$, then there exists a point $\alpha^* = (\alpha^*_1, \dots, \alpha^*_d) \in F^{-1}\beta$ such that*

$$\begin{aligned}\dim_H C^F_\beta &= \frac{1}{\log k} \max_{(\gamma_0,\dots,\gamma_{k-1}) \in \Gamma_k(\alpha^*)} \left\{ -\sum_{\ell=0}^{k-1} \gamma_\ell \log \gamma_\ell \right\} \\ &= \frac{1}{\log k} \inf_{(q_1,\dots,q_d) \in \mathbb{R}^d} \left\{ \log \sum_{\ell=0}^{k-1} \exp \sum_{j=1}^{d} q_j(\psi_{j\ell} - \alpha^*_j) \right\},\end{aligned} \tag{24}$$

   and there exists an ergodic nonlinear full measure ν_β for β.

Proof. Item 1 and the first equality in (24) follow directly from combining Theorem 10 in [6] (or Theorem 7.2 in [36]), Theorem 6 and Theorem 11 in [6].

The second equality in (24) follows readily from Theorem 6 using the particular expression for the topological pressure of 1-locally continuous functions in (23). Finally, the existence of nonlinear full measures is a consequence of the second item in Theorem 6. □

Remark 9. In view of Theorem 7.2 in [36] (or Theorem 10 in [6]), one can replace the hypothesis $F^{-1}\beta \cap L \subset \operatorname{int} L$ by the requirement that the linear dimension spectrum $\alpha \mapsto \dim_u C_\alpha$ restricted to $F^{-1}\beta$ attains its maximum at some point in $\operatorname{int} L$. This is useful for applications when $F^{-1}\beta \cap \partial L \neq \emptyset$ (see Example 6).

We note that Theorem 15 improves Theorem 10 in [6] and can be seen as a full nonlinear version of Theorem 11 in that paper. The main novelties are the existence of ergodic measures concentrated on the nonlinear level sets, and the second equality in (24), where we have a new more direct way of calculating the Hausdorff dimension. This new formula comes from the relation between the u-dimension and the classical topological pressure given by Theorem 6.

We continue to write $x = 0.x_1x_2\cdots$ in base k. Considering the number

$$\tau_m(x,n) = \#\{j \in \{0,\dots,n-1\} : x_j = m\}$$

with $m \in \{0,\dots,k-1\}$, we define the *frequency of* m in the base k-representation of x by

$$\tau_m(x) = \lim_{n\to\infty} \frac{\tau_m(x,n)}{n},$$

whenever the limit exists.

Example 6. Let $T_3x = 3x \bmod 1$ on $[0,1]$ and consider the level set

$$\mathcal{F}_\beta = \big\{x \in [0,1] : \tau_1(x) = f(\tau_0(x)) + \beta\big\},$$

where $f\colon \mathbb{R} \to \mathbb{R}$ is given by $f(z) = e^{1-4z}/4$ and $\beta \in \mathbb{R}$. Letting $\Psi = \{\psi_1, \psi_2\}$ with the 1-locally constant functions $\psi_1 = 1_{[0,1/3]}$ and $\psi_2 = 1_{[1/3,2/3]}$, we have that

$$\mathcal{F}_\beta = \big\{x \in [0,1] : G(\tau_0(x), \tau_1(x)) = \beta\big\} =: C^G_\beta,$$

where $G(x,y) = y - f(x)$. For each $(\alpha_1,\alpha_2) \in [0,1]\times[0,1]$, consider the level set

$$C_{(\alpha_1,\alpha_2)} = \big\{x \in [0,1] : \tau_0(x) = \alpha_1 \text{ and } \tau_1(x) = \alpha_2\big\}.$$

In this case, $L = \{(\alpha_1,\alpha_2) \in [0,1]\times[0,1] : \alpha_1+\alpha_2 \le 1\}$ (this set has nonempty interior) and

$$G^{-1}\beta = \left\{\left(\alpha, \frac{e^{1-4\alpha}}{4} + \beta\right) : \alpha \in \mathbb{R}\right\} \subset \mathbb{R}^2.$$

For each β with $G^{-1}\beta$ intersecting $\operatorname{int} L$, it is easy to see that $G^{-1}\beta$ intersects ∂L at exactly two points. Assume that $\beta \in \mathbb{R}$ is such that the maximum of the spectrum $(\alpha_1,\alpha_2) \mapsto \dim_H C_{(\alpha_1,\alpha_2)}$ restricted to $G^{-1}\beta$ is attained at some point $\alpha^* = (\alpha_1^*,\alpha_2^*) \in \operatorname{int} L$. It follows from Theorem 15 that

$$\begin{aligned}\dim_H \mathcal{F}_\beta = \dim_H C^G_\beta &= \max_{(\alpha_1,\alpha_2)\in G^{-1}\beta} \dim_H C_{(\alpha_1,\alpha_1)}\\ &= \frac{1}{\log 3} \inf_{(q_1,q_2)\in\mathbb{R}^2} \log \sum_{\ell=0}^{2} \exp \sum_{j=1}^{2} q_j(\psi_{j\ell} - \alpha_j^*).\end{aligned}$$

Since $\psi_{10} = 1, \psi_{11} = 0, \psi_{12} = 0, \psi_{20} = 0, \psi_{21} = 1$ and $\psi_{22} = 0$, we obtain

$$\dim_H \mathcal{F}_\beta = \frac{1}{\log 3} \inf_{(q_1,q_2)\in\mathbb{R}^2} \log\big(e^{q_1(1-\alpha_1^*)-q_2\alpha_2^*} + e^{q_2(1-\alpha_2^*)-q_1\alpha_1^*} + e^{-q_1\alpha_1^*-q_2\alpha_2^*}\big). \quad (25)$$

For instance, when $\beta = 0$, one can check that the maximum of the linear spectrum restricted to $G^{-1}0$ is attained at $\alpha^* = (1/4,1/4) \in \operatorname{int} L$. A computation using (25) gives that

$$\dim_H \mathcal{F}_0 = \dim_H C_{(1/4,1/4)} = \frac{3\log 2}{2\log 3} \approx 0.946395.$$

It follows from the proof of Theorem 6 that the ergodic nonlinear full measure $\nu_0 \in \mathcal{M}(T_3)$ for $\beta = 0$ is the unique equilibrium measure for the 1-locally continuous potential $\mathcal{H}_{\alpha^*}\colon [0,1] \to \mathbb{R}$ given by

$$\begin{aligned}\mathcal{H}_{\alpha^*}(x) &= \langle (q_1,q_2), (\psi_1 - \alpha_1^*, \psi_2 - \alpha_2^*)\rangle - \dim_H \mathcal{F}_0 \log 3\\ &= -\log 2(1_{[0,1/3]}(x) + 1_{[1/3,2/3]}(x) + 1).\end{aligned}$$

Since in particular (T_3, Ψ) is C^1 Legendre, Theorem 12 guarantees, a priori, the existence of at least one point $(\widetilde{\alpha}_1, \widetilde{\alpha}_2) \in \operatorname{int} L$ such that

$$\dim_H C_{(\widetilde{\alpha}_1,\widetilde{\alpha}_2)} = \dim_H [0,1] = 1.$$

Hence, for $\widetilde{\beta} \in \mathbb{R}$ such that $(\widetilde{\alpha}_1, \widetilde{\alpha}_2) \in G^{-1}\widetilde{\beta}$, we have

$$\dim_H \mathcal{F}_{\widetilde{\beta}} = \dim_H C^G_{\widetilde{\beta}} = \dim_H C_{(\widetilde{\alpha}_1,\widetilde{\alpha}_2)} = 1.$$

By the strict concavity of the linear dimension spectrum on L, the global maximum is unique, and in this case

$$(\widetilde{\alpha}_1, \widetilde{\alpha}_2) = \left(\frac{1}{3}, \frac{1}{3}\right) \quad \text{and} \quad \widetilde{\beta} = \frac{1}{12}\left(4 - \frac{3}{e^{1/3}}\right).$$

The graph of the spectrum $\beta \mapsto \dim_H \mathcal{F}_\beta$ can be obtained by a numerical computation using formula (25). The domain is given by

$$\left\{\beta \in \mathbb{R} : \max_{\alpha \in L \cap G^{-1}\beta} \dim C_\alpha = \max_{\alpha \in \operatorname{int} L} \dim C_\alpha\right\},$$

which is approximately the open interval $(-0.67, 0.5)$ (see Figure 5).

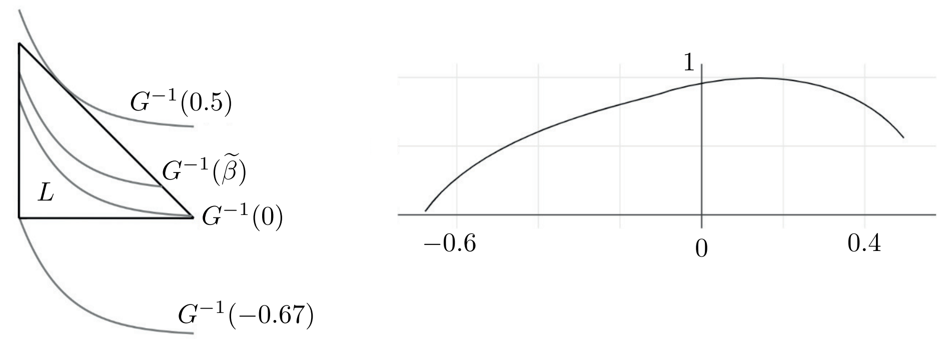

FIGURE 5. The domain and the graph of the nonlinear dimension spectrum $\beta \mapsto \dim_H \mathcal{F}_\beta$.

We are also able to show that the set of numbers with no well-defined nonlinear relations between frequencies has full Hausdorff dimension, despite being negligible from a measure theoretic point of view. The following result is a nonlinear version of Theorems 1 and 6 in [6].

Theorem 16. *For the expanding map T_k, consider a collection $\Psi = \{\psi_1, \ldots, \psi_d\}$ of 1-locally constant functions $\psi_j \colon [0,1] \to \mathbb{R}$ and let $F = (F_1, \ldots, F_p) \colon U \to \mathbb{R}^p$ be a continuous function, where U is an open set containing $L \subset \mathbb{R}^d$. If the function $F_j \colon U \to \mathbb{R}$ satisfies the void property with respect to (T_k, Ψ) for $j = 1, \ldots, p$, then*

$$\dim_H \left([0,1] / \bigcup_{\beta \in \mathbb{R}^p} C^F_\beta \right) = 1.$$

Proof. One can obtain a particular version of Theorem 13 for the map T_k and sequences of piecewise Hölder continuous functions with finitely many discontinuities and at most at the positive integer powers of $1/k$ (see Lemma 1 in [6]). In particular, this includes sequences of 1-locally constant functions and, using the ergodic nonlinear full measures given by Theorem 15, one can complete the argument proceeding as in the proof of Theorem 14. □

Example 7. Under the assumptions of Example 6, since each $G^{-1}\beta$ is a curve in L, the function G satisfies the void property with respect to (T_3, Ψ). Hence, it follows from Theorem 16 that the corresponding nonlinear irregular set with respect to G has full Hausdorff dimension, that is,

$$\dim_H\left([0,1]/\bigcup_{\beta\in\mathbb{R}}\mathcal{F}_\beta\right) = \dim_H \mathcal{I}_G = 1.$$

9. Relation to multiple ergodic averages

Here we explore some connections with the multifractal analysis of multiple ergodic averages. More precisely, using Theorem 6 and some results in [19], we compare different nonlinear dimension spectra and study the irregular sets for some types of multiple ergodic averages.

Let Σ be a finite set and consider the shift map $\sigma\colon \Sigma^{\mathbb{N}} \to \Sigma^{\mathbb{N}}$. Taking a collection of real-valued Hölder continuous functions $\Phi = \{\phi_1, \phi_2\}$ on $\Sigma^{\mathbb{N}}$ (such that the cohomology classes of $1, \phi_1, \phi_2$ are linearly independent), we define the *multiple ergodic average* level set

$$M_A(\beta) = \left\{x \in \Sigma^{\mathbb{N}} : \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\phi_1(\sigma^k(x))\phi_2(\sigma^{2k}(x)) = \beta\right\}.$$

The *invariant spectrum* of $M_A(\beta)$ is defined by

$$F_{\rm inv}(\beta) = \sup_{\mu\in\mathcal{M}_{\rm erg}(\sigma)}\{\dim_H \mu : \mu(M_A(\beta)) = 1\}.$$

When there exists an ergodic σ-invariant measure concentrated on $M_A(\beta)$, it follows from Theorem 1.5 in [19] that

$$F_{\rm inv}(\beta) = \sup_{\mu\in\mathcal{M}_{\rm erg}(\sigma)}\left\{\dim_H \mu : \int_{\Sigma^{\mathbb{N}}}\phi_1\,d\mu\int_{\Sigma^{\mathbb{N}}}\phi_2\,d\mu = \beta\right\}. \tag{26}$$

Now consider the function $F(x, y) = xy$. By Proposition 10 and Theorem 2, the linear spectrum

$$\alpha \mapsto \dim_H C_\alpha = \frac{1}{\log \#\Sigma}h_{\rm top}(T|C_\alpha)$$

attains its maximum on $\operatorname{int} L$. Since the spectrum is also concave on L, for each $\beta \in \mathbb{R}$ with $F^{-1}\beta\cap\operatorname{int} L \neq \emptyset$, the map $\alpha \mapsto \dim_H C_\alpha$ restricted to $F^{-1}\beta\cap L$ attains a maximal value at some point $\alpha^* \in \operatorname{int} L$. Hence, by Theorem 7.2 in [36] and Theorem 6, we have

$$\begin{aligned}\dim_H C^F_\beta &= \sup_{\alpha\in F^{-1}\beta}\dim_H C_\alpha = \dim_H C_{\alpha^*}\\ &= \dim_H \nu_\beta = \sup_{\mu\in\mathcal{M}_{\rm erg}(\sigma)}\left\{\dim_H \mu : F\left(\int_{\Sigma^{\mathbb{N}}}\phi_1\,d\mu, \int_{\Sigma^{\mathbb{N}}}\phi_2\,d\mu\right) = \beta\right\},\end{aligned}$$

where ν_β is an ergodic nonlinear full measure for β (with $\nu_\beta(C_{\alpha^*}) = 1$) and

$$C^F_\beta = \left\{x \in \Sigma^{\mathbb{N}} : \lim_{n\to\infty}\frac{1}{n^2}\sum_{k=0}^{n-1}\phi_1(\sigma^k(x))\sum_{k=0}^{n-1}\phi_2(\sigma^k(x)) = \beta\right\}.$$

Together wth (26), this implies that

$$\dim_H C^F_\beta = \dim_H \nu_\beta = F_{\rm inv}(\beta)$$

with ν_β concentrated on C^F_β. And since $F_{\rm inv}(\beta) \le \dim_H M_A(\beta)$ (see [19]), we obtain

$$\dim_H C^F_\beta = \dim_H \nu_\beta \le \dim_H M_A(\beta). \tag{27}$$

In addition, since (σ, Φ) is a C^1 Legendre pair, by Theorem 12 there is $\beta^* \in \mathbb{R}$ such that $\dim_H C^F_{\beta^*} = \dim_H \Sigma^{\mathbb{N}}$. Together with (27), this readily implies that $M_A(\beta^*)$ also has full Hausdorff dimension. Furthermore, considering the relation with the classical topological pressure given by Theorem 6, we obtain the following result.

Theorem 17. *Let $\sigma\colon \Sigma^{\mathbb{N}} \to \Sigma^{\mathbb{N}}$ be the shift map and let $\Phi = \{\phi_1, \phi_2\}$ be a collection of real-valued Hölder continuous functions on $\Sigma^{\mathbb{N}}$ such that the cohomology classes of $1, \phi_1, \phi_2$ are linearly independent. Then for each β with $F^{-1}\beta \cap \operatorname{int} L \neq \emptyset$ such that there is a measure $\mu \in \mathcal{M}_{\mathrm{erg}}(\sigma)$ with $\mu(M_A(\beta)) = 1$, the following properties hold:*

1. *there exists an ergodic σ-invariant measure ν_β concentrated on C^F_β such that*
$$\begin{aligned}\dim_H M_A(\beta) \geq F_{\mathrm{inv}}(\beta) &= \dim_H \nu_\beta = \dim_H C^F_\beta \\ &= \frac{1}{\log \#\Sigma} \max_{(\alpha_1,\alpha_2)\in F^{-1}\beta} \inf_{(q_1,q_2)\in\mathbb{R}^2} P(q_1\phi_1 + q_2\phi_2 - q_1\alpha_1 - q_2\alpha_2),\end{aligned}$$
where P is the classical topological pressure; in particular, there exists a point $(\alpha_1^*, \alpha_2^*) \in \operatorname{int} L \cap F^{-1}\beta$ such that
$$\dim_H M_A(\beta) \geq \frac{1}{\log \#\Sigma} \inf_{(q_1,q_2)\in\mathbb{R}^2} P(q_1\phi_1 + q_2\phi_2 - q_1\alpha_1^* - q_2\alpha_2^*);$$
2. *there exists at least one parameter $\beta^* \in F(L) \subset \mathbb{R}$ such that*
$$\dim_H M_A(\beta^*) = F_{\mathrm{inv}}(\beta^*) = \dim_H C^F_{\beta^*} = \dim_H \Sigma^{\mathbb{N}}.$$

Remark 10. In [19] the authors introduced a nonlinear-type topological pressure to calculate the exact value of the Haudorff dimension of the sets $M_A(\beta)$. By Theorem 17, one can see explicitly how the classical (linear) topological pressure is only able to give a lower bound in this framework. Moreover, making a comparison with [19], one can also obtain an expression connecting the classical and the "nonlinear" pressure introduced in that paper (see Theorem 1.1 in [19]).

In view of Theorem 17, the existence of nonlinear full measures and the notion of distinguishing measures, one can also study nontypical points with respect to some multiple ergodic averages. Let $\mathcal{I}_A$ be the set of points $x \in \Sigma^{\mathbb{N}}$ such that the limit $\lim_{n\to\infty} A_n(x)$ does not exist, where
$$A_n(x) = \frac{1}{n}\sum_{k=0}^{n-1} \phi_1(\sigma^k(x))\phi_2(\sigma^{2k}(x)).$$

Theorem 18. *Let $\sigma\colon \Sigma^{\mathbb{N}} \to \Sigma^{\mathbb{N}}$ be the shift map and let $\Phi = \{\phi_1, \phi_2\}$ be a collection of real-valued Hölder continuous functions on $\Sigma^{\mathbb{N}}$ such that the cohomology classes of $1, \phi_1, \phi_2$ are linearly independent. If each β with $F^{-1}\beta \cap \operatorname{int} L \neq \emptyset$ is such that there is a measure $\mu \in \mathcal{M}_{\mathrm{erg}}(\sigma)$ with $\mu(M_A(\beta)) = 1$, then*
$$\dim_H \mathcal{I}_A = \dim_H \Sigma^{\mathbb{N}}.$$

Proof. By the second item in Theorem 17, there exists $\beta^* \in F(L)$ that has associated an ergodic nonlinear full measure ν_{β^*} such that $\dim_H \nu_{\beta^*} = \dim_H \Sigma^{\mathbb{N}}$. One can verify that the supremum in (26) (with β replaced by β^*) is attained at some mixing measure $\eta_{\beta^*} \in \mathcal{M}_{\mathrm{erg}}(\sigma)$. By the mixing property, we have
$$\lim_{n\to\infty} A_n(x) = \int_{\Sigma^{\mathbb{N}}} \phi_1\, d\eta_{\beta^*} \int_{\Sigma^{\mathbb{N}}} \phi_2\, d\eta_{\beta^*} = \beta^* \quad \text{for } \eta_{\beta^*}\text{-almost every } x \in \Sigma^{\mathbb{N}}.$$
In particular, $\eta_{\beta^*}(M_A(\beta^*)) = 1$ and $F_{\mathrm{inv}}(\beta^*) = \dim_H \eta_{\beta^*}$. The first item in Theorem 17 now gives that $\dim_H \eta_{\beta^*} = \dim_H \nu_{\beta^*}$.

The function $F(x, y) = xy$ satisfies the void property with respect to (σ, Φ), and so for each $\varepsilon > 0$ there exists $\beta \in F(L)$ with $\beta \neq \beta^*$ such that $\dim_H \nu_\beta =$

$\dim_H \Sigma^{\mathbb{N}} - \varepsilon$ (see the proof of Theorem 14), where ν_β is an ergodic nonlinear full measure for β. Proceeding as above, one can also guarantee the existence of a measure $\eta_\beta \in \mathcal{M}_{\rm erg}(\sigma)$ concentrated on $M_A(\beta)$ and satisfying $F_{\rm inv}(\beta) = \dim_H \eta_\beta$. Again by the first item in Theorem 17, we obtain $\dim_H \eta_\beta = \dim_H \nu_\beta$. In this case, we also have

$$\lim_{n\to\infty} A_n(x) = \beta \quad \text{for } \eta_\beta\text{-almost every } x \in \Sigma^{\mathbb{N}},$$

and since $\beta \neq \beta^*$, the set $\{\eta_{\beta^*}, \eta_\beta\}$ is a collection of ergodic σ-invariant distinguishing measures for the sequence $(A_n)_{n\in\mathbb{N}}$. Therefore, Theorem 13 yields that

$$\begin{aligned}\dim_H \mathcal{I}_A &\geq \min\{\dim_H \eta_{\beta^*}, \dim_H \eta_\beta\}\\ &= \min\{\dim_H \nu_{\beta^*}, \dim_H \nu_\beta\} = \dim_H \Sigma^{\mathbb{N}} - \varepsilon.\end{aligned}$$

The arbitrariness of ε shows that $\dim_H \mathcal{I}_A = \dim_H \Sigma^{\mathbb{N}}$. □

To the best of our knowledge, Theorem 18 is the first result in the literature showing that some irregular sets for multiple ergodic averages have full Hausdorff dimension.

10. Intermediate entropy and dimension properties

In this section, inspired by the recent work in [17], we study some relations between intermediate entropy and dimension properties of ergodic measures and their nonlinear counterparts. Once more, these developments are based on the existence of ergodic nonlinear full measures as introduced in Section 3.1.

Let $\Phi = \{\phi_1, \ldots, \phi_d\}$ be a collection of real-valued continuous functions on X, and let $F\colon U \to \mathbb{R}^p$ be a continuous function, where U is an open set containing the domain L. For each $\beta \in F(L) \subset \mathbb{R}^p$, we define

$$h^F(\beta) = \sup_{\mu\in\mathcal{M}(T)} \left\{ h_\mu(T) : F\left(\int_X \Phi \, d\mu\right) = \beta \right\}.$$

Moreover, we continue to use the notations $\mathcal{E}^F$ and $\mathcal{D}_u^F$ introduced in Section 5.

The following result gives a relation between intermediate entropy properties of ergodic measures and their nonlinear versions.

Theorem 19. *Let $T\colon X \to X$ be a continuous map of a compact metric space such that the metric entropy $\mu \mapsto h_\mu(T)$ is upper semicontinuous. Let $\Phi = \{\phi_1, \ldots, \phi_d\}$ be a collection of real-valued continuous functions on X and assume that $F\colon U \to \mathbb{R}^p$ is a continuous function, where U is an open set containing L. If for all $\alpha \in \operatorname{int} L$ and $h \in [0, h(\alpha))$, the set*

$$\mathcal{B}(\alpha, h) = \left\{ \mu \in \mathcal{M}_{\rm erg}(T) : h_\mu(T) = h \text{ and } \int_X \Phi \, d\mu = \alpha \right\}$$

is a dense G_δ subset of

$$\mathcal{C}(\alpha, h) = \left\{ \mu \in \mathcal{M}(T) : h_\mu(T) \geq h \text{ and } \int_X \Phi \, d\mu = \alpha \right\},$$

then for all $\beta \in \mathbb{R}^p$ such that $F^{-1}\beta \cap L \subset \operatorname{int} L$ and $h' \in [0, h^F(\beta))$ the set

$$\mathcal{B}^F(\beta, h') = \left\{ \mu \in \mathcal{M}_{\rm erg}(T) : h_\mu(T) = h' \text{ and } F\left(\int_X \Phi \, d\mu\right) = \beta \right\}$$

is also a dense G_δ subset of

$$\mathcal{C}^F(\beta, h') = \left\{ \mu \in \mathcal{M}(T) : h_\mu(T) \geq h' \text{ and } F\left(\int_X \Phi \, d\mu\right) = \beta \right\}.$$

Proof. By Proposition 5.7 in [16], $\mathcal{M}_{\text{erg}}(T)$ is a G_δ subset of $\mathcal{M}(T)$. Therefore,

$$\left\{\mu \in \mathcal{M}_{\text{erg}}(T) : h_\mu(T) \geq h' \text{ and } F\left(\int_X \Phi\, d\mu\right) = \beta\right\}$$

is a G_δ subset of $\mathcal{C}^F(\beta, h')$. Since the metric entropy is upper semicontinuous, the set $\{\mu \in \mathcal{M}(T) : h_\mu(T) \in [h', h' + 1/n)\}$ is open in $\{\mu \in \mathcal{M}(T) : h_\mu(T) \geq h'\}$ for each $n \in \mathbb{N}$. This implies that

$$\left\{\mu \in \mathcal{M}(T) : h_\mu(T) = h' \text{ and } F\left(\int_X \Phi d\mu\right) = \beta\right\}$$

is a G_δ subset of $\mathcal{C}^F(\beta, h')$. Thus, the set $\mathcal{B}^F(\beta, h')$ is a G_δ subset of $\mathcal{C}^F(\beta, h')$. Finally, $\mathcal{B}^F(\beta, h')$ is dense in $\mathcal{C}^F(\beta, h')$, which completes the proof. □

As a consequence of Theorem 19 together with Theorems 3 and 6, we obtain a corresponding nonlinear result for intermediate entropy properties of ergodic measures.

Corollary 20. *Let $T\colon X \to X$ be a continuous map of a compact metric space such that the metric entropy $\mu \mapsto h_\mu(T)$ is upper semicontinuous. Let $\Phi = \{\phi_1, \dots, \phi_d\}$ be a collection of real-valued continuous functions on X and assume that $F\colon U \to \mathbb{R}^p$ is a continuous function, where U is an open set containing L. Assume that for all $\alpha \in \operatorname{int} L$ and $h \in [0, h(\alpha))$, the set $\mathcal{B}(\alpha, h)$ is a dense G_δ subset of $\mathcal{C}(\alpha, h)$. Then for each $\beta \in \mathbb{R}^p$ with $F^{-1}\beta \cap L \subset \operatorname{int} L$ we have*

$$[0, h^F(\beta)) \subset \left\{h_\mu(T) : \mu \in \mathcal{M}_{\text{erg}}(T) \text{ and } F\left(\int_X \Phi\, d\mu\right) = \beta\right\}.$$

Moreover, if there is a dense subspace $D(X) \subset C^0(X)$ such that every $\xi \in D(X)$ has a unique equilibrium measure with respect to T, then for each $\beta \in \mathbb{R}^p$ with $F^{-1}\beta \cap L \subset \operatorname{int} L$ we have

$$[0, \mathcal{E}^F(\beta)) \subset \left\{h_\mu(T) : \mu \in \mathcal{M}_{\text{erg}}(T) \text{ and } F\left(\int_X \Phi d\mu\right) = \beta\right\}.$$

In particular, when $h_{\text{top}}(T) < \infty$ and $\operatorname{span}\{\phi_1, \dots, \phi_d, u\} \subset D(X)$, for each $\beta \in \mathbb{R}^p$ with $F^{-1}\beta \cap L \subset \operatorname{int} L$ we have

$$\begin{aligned}[0, \mathcal{E}^F(\beta)] &= \left\{h_\mu(T) : \mu \in \mathcal{M}_{\text{erg}}(T) \text{ and } F\left(\int_X \Phi d\mu\right) = \beta\right\}\\ &= \left\{h_\mu(T) : \mu \in \mathcal{M}(T) \text{ and } F\left(\int_X \Phi d\mu\right) = \beta\right\}.\end{aligned}$$

Now assume that $T\colon X \to X$ is a homeomorphism. We say that T is *expansive* if there exists a constant $c > 0$ such that for any $x \neq y \in X$, we have $d(T^i(x), T^i(y)) > c$ for some $i \in \mathbb{Z}$. Given $\delta > 0$, a sequence $\{x_n\}_{n\in\mathbb{Z}}$ is called a *δ-pseudo-orbit* if $d(T(x_n), x_{n+1}) < \delta$ for any $n \in \mathbb{Z}$. Moreover, given $\varepsilon > 0$, we say that $\{x_n\}_{n\in\mathbb{Z}}$ is *ε-shadowed* by some $y \in X$ if $d(T^n(y), x_n) < \varepsilon$ for any $n \in \mathbb{Z}$. Finally, we say that the map T has the *shadowing property* if for each $\varepsilon > 0$ there exists $\delta > 0$ such that any δ-pseudo-orbit is ε-shadowed by some point in X.

Following [17], we say that a homeomorphism $T\colon X \to X$ of a compact metric space is *topologically hyperbolic* if X has infinitely many points, and T is expansive and satisfies the shadowing property. It is well known that locally maximal hyperbolic sets for C^1 diffeomorphisms are expansive and satisfy the shadowing property (see for example [24]). It follows from Theorem 6.11 (III) in [17] that if T is a transitive topologically hyperbolic system, then all the hypotheses of Corollary 20 are satisfied.

Following the proof of Corollary C in [17] (with simple modifications), we obtain the following result for the u-dimension.

Theorem 21. *Suppose that $T\colon X \to X$ is topologically transitive and topologically hyperbolic, and let $\Phi = \{\phi_1, \ldots, \phi_d\}$ be a collection of real-valued continuous functions on X. Then for each $\alpha \in \operatorname{int} L$ the set*

$$\left\{\dim_u \mu : \mu \in \mathcal{M}_{\text{erg}}(T) \text{ and } \int_X \Phi \, d\mu = \alpha\right\}$$

is an interval containing zero, that is,

$$[0, \Delta_u^{\text{erg}}(\alpha)) \subset \left\{\dim_u \mu : \mu \in \mathcal{M}_{\text{erg}}(T) \text{ and } \int_X \Phi \, d\mu = \alpha\right\} \subset [0, \Delta_u^{\text{erg}}(\alpha)],$$

where $\Delta_u^{\text{erg}}(\alpha) = \sup\bigl\{\dim_u \mu : \mu \in \mathcal{M}_{\text{erg}}(T) \text{ and } \int_X \Phi \, d\mu = \alpha\bigr\}$.

Combining Theorem 21 with Theorems 3 and 6, we obtain a corresponding nonlinear result for the intermediate u-dimension properties of ergodic measures.

Corollary 22. *Suppose that $T\colon X \to X$ is topologically transitive and topologically hyperbolic, and let $\Phi = \{\phi_1, \ldots, \phi_d\}$ be a collection of real-valued continuous functions on X. Then for each $\beta \in \mathbb{R}^p$ with $F^{-1}\beta \cap L \subset \operatorname{int} L$ we have*

$$[0, \Delta_{u,F}^{\text{erg}}(\beta)) \subset \left\{\dim_u \mu : \mu \in \mathcal{M}_{\text{erg}}(T) \text{ and } F\left(\int_X \Phi \, d\mu\right) = \beta\right\},$$

where

$$\Delta_{u,F}^{\text{erg}}(\beta) = \sup\left\{\dim_u \mu : \mu \in \mathcal{M}_{\text{erg}}(T) \text{ and } F\left(\int_X \Phi \, d\mu\right) = \beta\right\}.$$

Moreover, if there is a dense subspace $D(X) \subset C^0(X)$ such that every $\xi \in D(X)$ has a unique equilibrium measure with respect to T, then for each $\beta \in \mathbb{R}^p$ with $F^{-1}\beta \cap L \subset \operatorname{int} L$ we have

$$[0, \mathcal{D}_u^F(\beta)) \subset \left\{\frac{h_\mu(T)}{\int_X u \, d\mu} : \mu \in \mathcal{M}_{\text{erg}}(T) \text{ and } F\left(\int_X \Phi \, d\mu\right) = \beta\right\}.$$

In particular, when $h_{\text{top}}(T) < \infty$ and $\operatorname{span}\{\phi_1, \ldots, \phi_d, u\} \subset D(X)$, for each $\beta \in \mathbb{R}^p$ with $F^{-1}\beta \cap L \subset \operatorname{int} L$ we have

$$\begin{aligned}[0, \mathcal{D}_u^F(\beta)] &= \left\{\dim_u \mu : \mu \in \mathcal{M}_{\text{erg}}(T) \text{ and } F\left(\int_X \Phi \, d\mu\right) = \beta\right\} \\ &= \left\{\frac{h_\mu(T)}{\int_X u \, d\mu} : \mu \in \mathcal{M}(T) \text{ and } F\left(\int_X \Phi \, d\mu\right) = \beta\right\}.\end{aligned}$$

For average conformal hyperbolic maps, similar intermediate Hausdorff dimension properties for the nonlinear case can also be obtained from Corollary 22 (see Theorem G in [17]).

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